THE PROCEEDINGS OF THE PHYSICAL SOCIETY

Section B

VOL. 62, PART 5 1 May 1949 No. 353 B

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Composite subscription for both Sections A and B: £9 9*s*.

Published by
THE PHYSICAL SOCIETY
1 Lowther Gardens, Prince Consort Road, London S.W.7

PROCEEDINGS OF THE PHYSICAL SOCIETY

The *Proceedings* is now published monthly in two Sections.

Papers for publication in the *Proceedings* should be addressed to the Hon. Papers Secretary, Dr. H. H. HOPKINS, at the Office of the Physical Society, 1 Lowther Gardens, Prince Consort Road, London S.W.7. Telephone : KENsington 0048, 0049.

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PROCEEDINGS OF THE PHYSICAL SOCIETY

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THE PROCEEDINGS OF THE PHYSICAL SOCIETY

Section B

VOL. 62, PART 5 1 May 1949 No. 353 B

Vibrations of Free Rectangular Plates

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MS. received 19*th November* 1948

ABSTRACT. Records are given of the normal vibrating modes and frequencies of free rectangular plates between the limiting shapes of the bar and the square. The nodal systems, which in general consist of straight lines parallel to the sides, are, from considerations of symmetry, divided into four classes. Combined modes, for which the nodal patterns are less simple, are not uncommon. The constituent modes belong to the same class, but their uncombined periods may be appreciably different. The combination of modes belonging to different classes is extremely rare, the uncombined periods differing very little in frequency. As the mirror symmetry of the nodal design is lost in such combinations, it may be questioned whether they are ideally possible even for modes of exactly equal period.

§ 1. INTRODUCTION

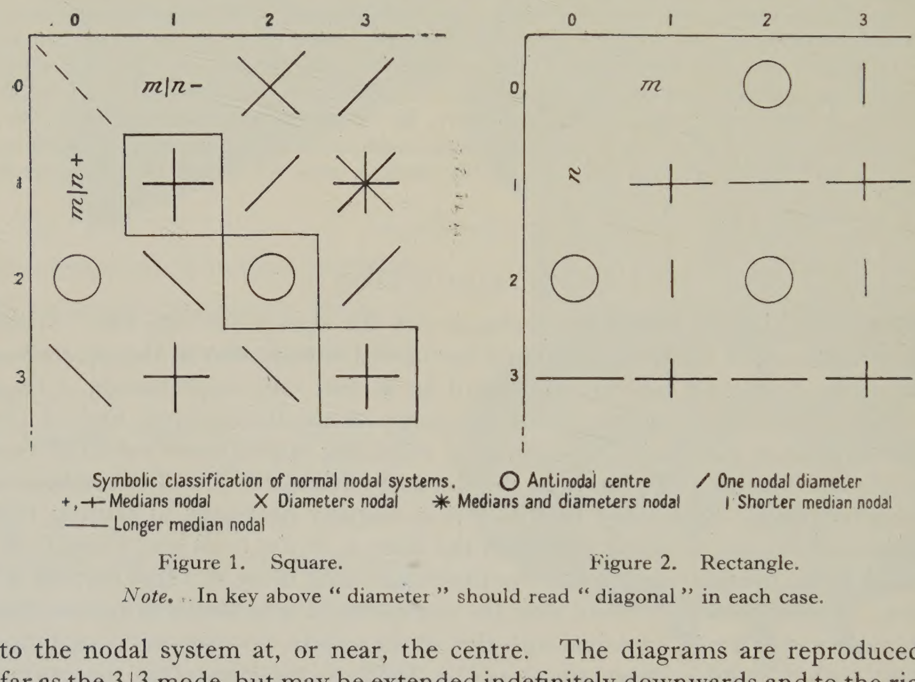
THE limiting shapes of the rectangle are the square and the bar. While the theory of the freely vibrating bar is well known, that of the square has been developed more recently and is, as yet, only approximate. Thus Rayleigh (1894, 1911) calculated the frequency of the fundamental tone of the vibrating square plate, and his method of obtaining approximate solutions (see Temple and Bickley 1933) has been developed by Ritz (1909) to obtain solutions for higher tones. According to Ritz " it is scarcely necessary to remark that the solution for the rectangle will be of the form $u_m(x/a)u_n(y/b) \pm u_n(x/a)u_m(y/b)$, a and b being the half-sides " and the functions being those of a free bar, see § 2 below. It is, however, evident that the symmetry of a rectangle is not, as this statement would imply, identical with that of the square, nor does such a solution accord with Chladni's observations (1787, 1802) to the effect that the normal nodal systems of rectangular plates consist, in general, of straight lines parallel to the sides—a conclusion which has been confirmed by certain observations and calculations of Pavlík (1936, 1937).

It is evidently desirable that full experimental data should be available, and the present work, nearly completed in 1939 and finished now with partly new equipment, gives photographic records of the nodal systems of various shaped rectangles together with experimentally determined data on the natural frequencies. As in a previous study on square plates (Waller 1939 b), the subject has been approached through considerations of symmetry with a view to classification and to obtaining also an understanding of the more complicated designs which sometimes occur when modes of nearly equal period combine.

The solid carbon dioxide sublimation method (Waller 1938, 1939b, 1941), supplemented occasionally by the use of the bow, has again been used to excite the plates, while the vibration frequencies have been determined as before by means of a calibrated valve-operated mains oscillator.

§ 2. SYMMETRY AND CLASSIFICATION

The symmetry of a rectangle, the sides of which are not equal, is less than that of the square. The latter possesses 90° rotational, the former only 180° rotational symmetry, and whereas there is mirror symmetry about both medians and diagonals in the nodal designs for the square, in the case of the rectangle the diagonal mirror symmetry is lost. It follows from the principle of symmetry (Rayleigh 1894, § 229) that while for the square as many as four nodal lines (medians and diagonals) may pass through the centre, for the rectangle it is not possible for more than two nodal lines (medians) to pass through its centre. This conclusion is fundamental, and Figures 1 and 2 exhibit the various classes into which the nodal systems of the square and rectangle may be divided by reference



Symbolic classification of normal nodal systems. ○ Antinodal centre / One nodal diameter
+, —+— Medians nodal X Diameters nodal ✳ Medians and diameters nodal | Shorter median nodal
—— Longer median nodal

Figure 1. Square. Figure 2. Rectangle.

Note. In key above " diameter " should read " diagonal " in each case.

to the nodal system at, or near, the centre. The diagrams are reproduced as far as the 3 | 3 mode, but may be extended indefinitely downwards and to the right; their meaning is made clear by the explanation of the descriptive abbreviation symbols given immediately below them. A summary of Figure 2, showing that rectangular nodal designs fall into four classes, is given in Table 1.

It will be seen that Figure 1 corresponds with the Rayleigh–Ritz approximate equation (Rayleigh 1894 and 1911, Ritz 1909):

$$\omega = u_m(x)u_n(y) \pm u_n(x)u_m(y) = 0, \qquad \ldots\ldots (1)$$

in which ω denotes the displacement at the point x, y, and $u(x)$, $u(y)$ are the normal functions of the transverse vibrations of a free bar which is equal in length to the side of the plate. In this equation two terms only of the Ritz series equation are retained.

In addition to these figures it is theoretically possible, but only when $m+n$ is odd, for nodal figures to exist according to the equation

$$\omega = Au_m(x)u_n(y) \pm Bu_n(x)u_m(y) = 0, \qquad \ldots\ldots (2)$$

where the ratio of the amplitude constants may have any value.

The physical meaning of the numbers m and n is made evident from the consideration that when B is zero the nodal systems consist of m lines parallel to the y axis and n lines parallel to the x axis, the origin being taken at the lower left-hand corner of the plate.

The approximate equation for the rectangle is accordingly

$$\omega = u_m(x/a)u_n(y/b) = 0, \qquad \ldots\ldots (3)$$

which, when either n or m is zero, represents straight lines parallel either to the shorter side b or to the longer side a of the plate. In the above equations m and n may have any integral values provided that their sum is not less than two.

Table 1. The Four Classes of Nodal Symmetry of Free Rectangular Plates

Specification of vibrating mode $m\|n$	Condition at centre	Details of nodal lines	Symbol
e|e	Antinode		○
o|o	Node	Both medians	$-\!\!+\!\!-$
e|o	Node	Longer median	——
o|e	Node	Shorter median	|

e denotes that the values of m and n are even, and o that they are odd (see Figure 2).

As is usual, the abbreviated notation $m|n+$ and $m|n-$ may be used to denote the square nodal systems occupying the left lower and right upper triangular spaces respectively in Figure 1. The notation $m|n$ will include all the normal rectangular modes of Figure 2.

§3. THE NORMAL NODAL FIGURES

Photographic records of the nodal designs obtained on narrow, medium and wide rectangles are shown in Figures 3–9 (Plates I and II). The prevalent design undoubtedly consists of lines parallel to the sides; note for example the 5|0, 4|4 and 7|3 higher modes in Figure 6.

The straightness of the lines formed on narrow rectangles may be seen in Figure 3, and the close connection between bars and narrow rectangles is emphasized in the 17|0 and 7|1 photographs of this group.

Anticlastic curvature, due to Poisson's ratio not being zero, is well exemplified in the 2|0 and 0|2 modes, and the pronounced curvatures, either convex or concave to the centre, which occur in the nodal lines as the width of the plate increases may be observed in Figures 4, 5 and 7. Finally, on the square, as we know, these figures assume the 2|0 − and 2|0 + forms respectively, in one of which the two diagonals are nodal, while in the other the antinodal centre is surrounded by a closed figure.

Rectangles may conveniently be referred to in terms of the approximate ratio of length to breadth, and the normal designs of 2, 3/2 and 1·09 rectangles are exhibited in Figures 4, 5 and 7 respectively. The systematic arrangement

is that of Figure 2, and the reader may in the first place study all the photographs for one rectangle, and afterwards take any particular mode and see how its nodal design varies as the ratio of length to breadth changes. Departures from the typical straight line designs are not uncommon, but their consideration will be deferred until the subject of compounded vibrations is dealt with in § 5.

§ 4. VIBRATION FREQUENCIES

(i) *The Natural Frequencies*

The natural frequencies of a rectangular plate are proportional to $(1/a^2)(D/m)^{\frac{1}{2}}$, where a is the length of the side, m is the mass per unit area and $D = Eh^3/12(1-\sigma^2)$ is the flexural rigidity; E is Young's modulus, h the thickness, and σ Poisson's ratio (Timoshenko 1937). Pavlík (1936, 1937), using the Rayleigh–Ritz approximate method, gives 61·4 kc/s. calculated, as compared with 60·0 kc/s. observed for the 1 | 1 mode of a stainless steel 3/2 plate of dimensions 9·034 mm. × 6·024 mm. × 1·025 mm. For comparison with brass it is convenient to note that a change in Poisson's ratio from $\sigma = 1/3$ (brass) to $\sigma = 1/4$ (steel) increases the fundamental frequency by about 4·8% (Lemke 1928). As regards differences in Young's modulus and density, the frequency of a brass plate should be about 7/10 that of a steel plate of the same dimensions.

(ii) *Relative Frequencies*

The frequencies (expressed in terms of the gravest tone taken as unity) corresponding to the classified nodal systems of Figures 4, 5 and 7 respectively, are given in Table 2, in which are also included particulars of the dimensions and actual lowest frequencies of some of the plates used. It will be noted that for identical values of m and n, the frequencies of the $m|n$ are less than those of the $n|m$ modes, a result to be expected, since the length exceeds the breadth of the plate. It will be seen also that the order of increasing frequency, as compared with the associated nodal figure, varies with the shape of plate; in particular that the gravest tone of a narrow rectangle has a 2 | 0, while that of a wide rectangle has a 1 | 1 nodal design.

(iii) *The $m|o$, $o|n$ Modes; Comparison with Bar; Note on Technique*

When the vibrations comprise only lines parallel to one side, the frequencies may be expected to bear some relation to those of a bar. How very close this relation is may be seen in Table 3, where the frequencies of the $m|o$ tones for the bar and various rectangles are given relatively to the 2 | 0 tone taken as unity. It will be noted that the $o|n$ sequence also is nearly that of the bar. Chladni's work on this subject will be mentioned later, § 5 (ii).

In this connection Doerffler's observations (1930) of high $m|o$ overtones produced on quartz plates by the piezoelectric method, are of interest; it would appear that such excitation is likely to favour these specialized vibrations. Dye (1932) excited quartz plates by the same means and used a stroboscopic interferometer method to obtain the figures, and now Tolansky's researches by multiple beam interferometry (1948) include among them a study of the piezoelectric oscillations of quartz crystals.

On account of the much higher frequencies produced, the spacing of nodal lines is much closer than for vibrations obtained by means of solid carbon dioxide

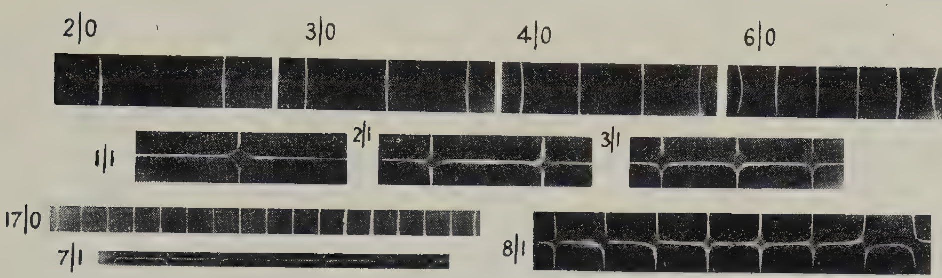


Figure 3

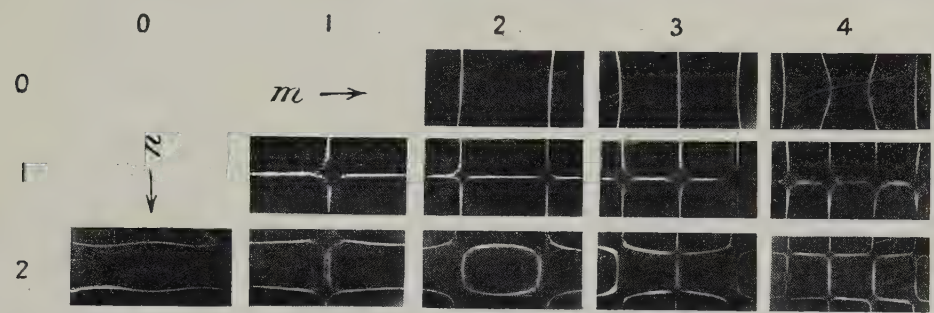


Figure 4.

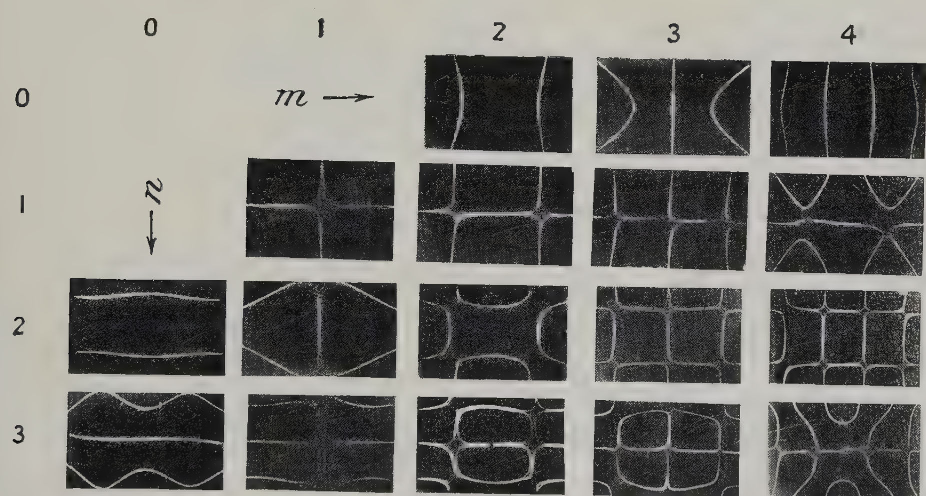


Figure 5.

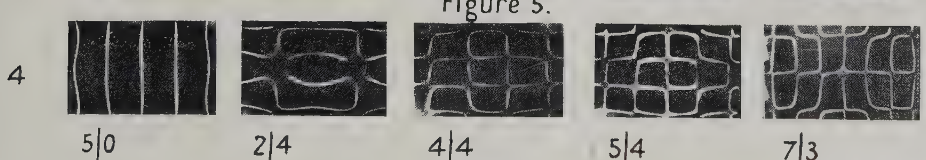


Figure 6.

Normal Nodal Systems of Free Rectangular Plate

Plate I.

To face page 280

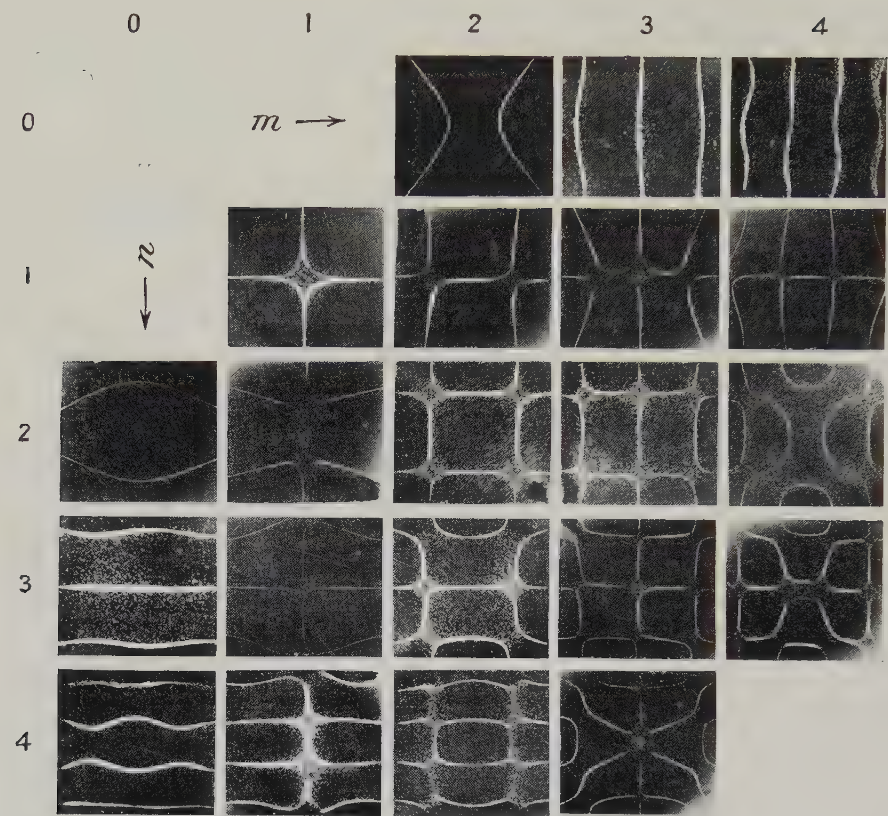


Figure 7.

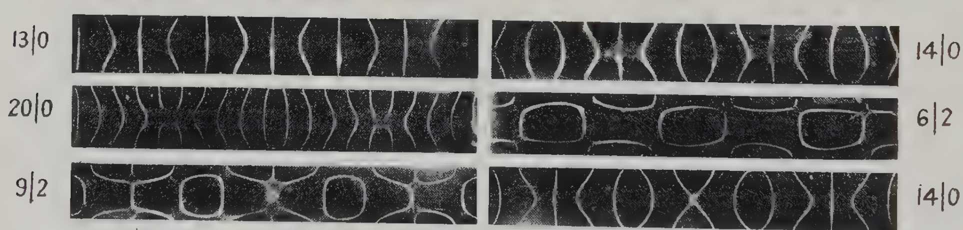


Figure 8.

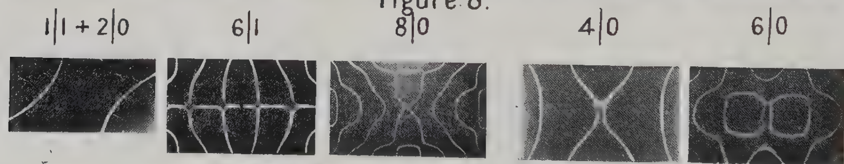


Figure 9.

Plate II.

where the frequencies are usually of the order 1,500 to 6,000 c/s. and the method, if it can be developed to produce truly free vibrations, might prove more effective in the detection of flaws. The solid carbon dioxide sublimation method is, however, unsurpassed for producing powerful and exceptionally free vibrations

Table 2. Relative Frequencies of Normal Vibrating Modes of Free Rectangular Plates (see Figures 4, 5 and 7)

2/1 Rectangle

	0	1	2	3	4
0			1*	2·88	5·42
1		1·20	2·30	3·62	*6·2*
2	4·37	4·87	6·7	8·2	10·8

* Actual frequency of 6·00 cm. × 2·98 cm. × 0·205 cm. plate was 1730 c/s. and of 14·1 cm. × 7·05 cm. × 0·315 cm. plate was 482 c/s.

1·09 Rectangle

	0	1	2	3	4
0			1·53	4·55	9·3
1		1*	2·67	5·16	*10*
2	2·23	2·78	5·1	*8*	*13*
3	5·35	6·4	8·8	11·5	*16*
4	10·8	11·8	*14*	*17*	*21*

* Actual frequency of 10·00 cm. × 9·20 cm. × 0·183 cm. plate was 423 c/s. and of 15·6 cm. × 14·4 cm. × 0·23 cm. plate was 220 c/s.

3/2 Rectangle

	0	1	2	3	4	5
0			1·08	2·93	5·53	9·96
1		1*	2·49	4·47	7·09	11·0
2	2·62	3·42	5·0	7·60	10·5	14·9
3	7·5	7·9	9·6	12·3	15·5	*20*
4	13·6	14·4	16·5	19·3	22·7	27

* Actual frequency of 24·9 cm. × 16·2 cm. × 0·237 cm. plate was 134 c/s.; smaller plates were also used.

The figures in italics are approximate.

Table 3. Comparison with Bar

Shape of rectangle	Vibrating mode 2|0	3|0	4|0	5|0
Bar*	1	2·76	5·40	8·93
12	1	2·76	5·37	8·9
6·32	1	2·74	5·40	
4	1	2·80	5·48	9·1

Shape of rectangle	Vibrating mode 2|0	3|0	4|0	5|0
2·08	1	2·88	5·42	
1·5	1	2·7	5·2	9·2
1·09	1	2·97	6·1	
	0|2	0|3	0|4	
1·5	1	2·8	5·2	

* See Barton (1908). The relative frequencies vary as $3{\cdot}011^2, 5^2, 7^2, \ldots$

in metal objects (and even in quartz (Waller 1933)), and has proved itself to be of great value for obtaining comprehensive records of the normal and combined vibrating modes of plates of any specified shape.

(iv) *Comparison of Results*

The comparison of results (see Table 4) is, in the interests of brevity, restricted to the $m=n$ tones, the nodal systems of which extend diagonally down from left to right in the classified figures.

Beginning with narrow rectangles and ending with the square, it will be seen that the present observations are in fair agreement with the vibration frequencies given by Chladni.

Pavlík's calculations, which, however, do not include narrow rectangles or the higher modes of the 1·5 rectangle, are in better agreement with the present results than his measured frequencies. With regard to the latter, it is to be noted that the lengths of his plates were only about ten times their thicknesses and that the Rayleigh–Ritz theory applies only to plates in which the thickness is small in comparison with the distance between adjacent nodal lines.

Table 4. Comparison of Results. Relative Frequencies

*l/b**	1|1	2|2	3|3	4|4	*l/b*	1|1	2|2	3|3	4|4
	Present observations					Chladni †			
6·3	1	9·7							
4	1	7·2			4	1	7·6		
2	1	6·7	10·8		2	1	5	12·5	
1·5	1	5	12·3	22·7					
1·09	1	4·9	11·5	21	1·12	1	4·75	11·3	20·6
1	1	4·8	11·8	21·5	1	1	4·8	10·7	20·2
	Pavlík, observed					Pavlík, calculated			
1·5	1	4·51			1·5	1	5·12		
1·07	1	4·42	9·84		1·07	1	4·83	11·4	
1	1	4·34	9·33		1	1	4·63	11·2	

* *l/b*=ratio of length to breadth of rectangle.

† See Zamminer's table (Rayleigh 1894, § 20) for converting chromatic scale notes to vibration frequencies. The relative frequencies given are necessarily approximate.

§ 5. COMPOUNDED NORMAL MODES

(i) *Combination of Modes of the same Class*

While the 2|2 nodal system of the wide rectangle, Figure 7, is normal, yet for the two narrower rectangles, Figures 4 and 5, the lines of this same mode are curved. This is because a second mode of small amplitude, 4|0, has combined with the principal vibration. The relative frequencies for Figure 4 are 6·7 and 5·42, and for Figure 5 are 5·0 and 5·53 respectively. Consider a second case of combined modes, which occurs in Figure 5, where the 4|1 mode contains a trace of the 0|3, while the 0|3 contains a trace of the 4|1 pattern, the relative frequencies being 7·09 and 7·5 respectively.

Now in order that two modes may combine it is theoretically necessary for the two periods to be equal. Though any friction arising from internal or external causes makes it possible for modes of slightly different period to combine, it is surprising that the periods can actually differ as much as this; furthermore, the greater prevalence of the compounded modes on rectangles as compared with the square also requires some explanation.

It will be noted that in the first example the centres of the combining modes are both antinodal. In the second example the longer median is nodal, the mirror symmetry of the compounded pattern about this median remaining unchanged. When other curved-line designs occurring in Figures 4, 5 and 7 are examined, and a list made as in Table 5, it becomes apparent that the modes which combine always belong to the same class.

Table 5. Compounded Normal Modes of Rectangle

<table>
<tr><th>l/b</th><th>Combining modes</th><th>Combining classes</th><th>Symbol</th></tr>
<tr><td>2</td><td>2|2, (4|0)</td><td>e|e</td><td>○</td></tr>
<tr><td></td><td>3|2, (1|4)</td><td>o|e</td><td>¦</td></tr>
<tr><td>1·5</td><td>2|2, (4|0)</td><td>e|e</td><td>○</td></tr>
<tr><td></td><td>4|1, (0|3)</td><td>o|o</td><td>—+—</td></tr>
<tr><td></td><td>0|3, (4|1)</td><td>e o</td><td>——</td></tr>
<tr><td></td><td>3|0, (1|2)</td><td>o|e</td><td>|</td></tr>
<tr><td></td><td>1|2, (3|0)</td><td>o|e</td><td>|</td></tr>
<tr><td></td><td>4|3, 6|1</td><td>e|o</td><td>——</td></tr>
<tr><td>1·09</td><td>4|2, (6|0)</td><td>e|e</td><td>○</td></tr>
<tr><td></td><td>2|3, (4|1)</td><td>e|o</td><td>——</td></tr>
</table>

The brackets indicate the weaker vibration.

Bearing in mind this significant conclusion we may turn again to the classification diagrams (Figures 1 and 2) and note that while the symmetry of the rectangle is repeated for every fourth mode, it is otherwise for the square, where the symmetry of the nodal designs occupying the right-hand upper triangular space do not in general resemble those occupying the left-hand lower triangular space.

[It may at first sight appear from the double-term equation (1) that figures on square plates are made up of two figures, but these "single" figures cannot exist separately (unless $m+n$ is odd) since Poisson's ratio is never zero, and compounded square modes are totally different (Waller 1940).]

(ii) *Chladni's Figures*

The subject cannot be left without considering Chladni's work on rectangular plates (1802). Tables of natural frequencies occur in the text, but the drawings are almost exclusively concerned with the way in which one nodal

Table 6. Chladni's Figures

Die Akustik, Figures 157–177, §§ 122–130.

<table>
<tr><th>Shape of rectangle</th><th>Combining modes</th><th>Shape of rectangle</th><th>Combining modes</th></tr>
<tr><td>6/5</td><td>4|1, 2|3</td><td>3/2</td><td>4|1, 0|3</td></tr>
<tr><td>5/4</td><td>5|0, 1|4</td><td>5/3</td><td>3|0, 0|2*†</td></tr>
<tr><td></td><td>0|4, 4|2</td><td>7/4</td><td>4|0, 2|2</td></tr>
<tr><td></td><td>4|1, 2|3</td><td>2/1</td><td>2|1, 3|0*</td></tr>
<tr><td>7/5</td><td>4|0, 0|3*†</td><td>7/3</td><td>4|0, 0|2†</td></tr>
<tr><td></td><td>1|2→3|0</td><td>3/1</td><td>5|0, 0|2*†</td></tr>
<tr><td></td><td>3|0→1|2</td><td></td><td></td></tr>
<tr><td></td><td>3|3→5|1</td><td></td><td></td></tr>
</table>

* Combination of "bar" modes. † Combination of mixed classes.

design can be modified and merged into another as a result of holding and exciting the plate at different places. Such manipulation introduces damping and constraint which favours the combination of tones. Information contained in the text is summarized in Table 6. We find, as expected, that most of the combinations belong to the same class, but there are four examples of the association of modes which belong to different classes, three of which are concerned with "bar" modes § 4 (iii). Chladni's interest in this subject was such that he cut his bars to shapes such as 7/5, where it may be expected that the 4|0 and 0|3 modes may combine since $7^2/5^2 \times 5^2/7^2 = 1$. He assuredly proved that the sequence of *m* *o* and *o* *n* frequencies was nearly that of the bar. His treatment of the rectangle as compared with some of the other shapes he studied is further evidence of the prominent part which combined modes assume on this particular shape of plate. It is noteworthy that in cases of mixed combination the mirror symmetry about the appropriate line through the centre is impaired.

(iii) *Combination of Mixed Classes. Gravest Tone of Rectangle*

The natural frequencies of two modes belonging to different classes are unlikely to be exactly equal, and it has been seen that in fact the combination of two such modes is not generally met with when the vibrations are free and the damping small. The rectangle, on account of its variable shape, is evidently suitable for studying the combination of modes, the ratio of whose periods can be gradually adjusted to equality by gently filing the edge of the plate.

An interesting case to study is the gravest vibration, which, as already mentioned, is associated on narrow rectangles with a 2|0, and on wide rectangles with a 1|1 nodal design. It has been found (Waller 1939a) that for brass the periods become equal when the ratio of length to breadth is 1·93, and a resulting nodal pattern is shown in the first photograph of the Figure 9 group. The resonance is very sharp and the pattern reverts to the 2|0 or 1|1 form as soon as the frequencies vary by a few cycles per second. It is particularly to be noted that in the combined design of this mixed-mode combination the mirror symmetry about the median through the centre has been destroyed. According to Rayleigh (1894), "in order that two modes of vibration may combine, it is only necessary that the periods agree". Yet in the case of combined modes belonging to different classes the mirror symmetry is impaired and one cannot but question whether such a combination is ideally possible.

(iv) *Distorted Figures*

Apart from distortions discussed in connection with Chladni's manipulated plates, it is not uncommon to meet with a more symmetrical distorted pattern. This generally consists of a vibration for which the centre would be antinodal were it not for the rigid central support—see, for example, the 14|0 photograph of a 6·32 rectangle in Figure 8, or the 8|0, 4|0, 6|0 photographs of a 3/2 rectangle in Figure 9.

(v) *Recognition of Nodal Designs*

The recognition of normal nodal designs presents no difficulties, but it is worth noting how they may sometimes be sub-divided. In Figure 8, for example, the 6|2 pattern is evidently made up of three 2|2, and the 9|2 of three 3|2 patterns. This division of a nodal design can be made whenever *m* or *n* (values greater than two) are divisible by an integer. Compounded modes are also not

difficult to recognize since they belong to the same class, so that the sum $m+n$ generally remains constant or occasionally varies by two (Table 5). In most cases one of the modes preponderates and this aids recognition of the second. The significance of wave-like lines, so familiar on circular and square plates, can be understood by examining, in Figure 6, the 4|1 (0|3) and 0|3 (4|1) designs. Each "wave" counts as two in the weaker vibration.

Finally it may again be emphasized that a nodal arrangement through the centre which does not fall into one of the four nodal classes denotes that the figure is distorted. As already stated, a common form of symmetrical distortion occurs when a figure of antinodal centre is affected by the central rigid support.

§6. CONCLUSION

Although the ratio of length to breadth of a rectangle may have any value, the experiments which have been made on narrow, medium and wide rectangles, respectively, may be considered to be reasonably comprehensive. The connection with bar-vibrations on the one hand and with square-vibrations on the other hand has been discussed. Considerations of symmetry have shown that the nodal systems can be divided into four classes according as the centre is antinodal or has passing through it one or both of the nodal median lines.

While the normal nodal systems consist of straight lines parallel to the sides (except for effects of anticlastic curvatures), compounded designs consisting chiefly of curved lines are not uncommon. It is significant that these are always made up of nodal systems belonging to the same class, and that the symmetry about the appropriate line through the centre remains unaltered. The periods may be appreciably different, and one of the modes usually has a considerably larger amplitude than the other.

In rare cases when the periods are sensibly equal, vibrations belonging to different classes combine. The resonance is then very sharp. Since, however, the former mirror symmetries are destroyed in the resulting figure, it is suggested that, despite equality of period, such combination is not ideally possible.

ACKNOWLEDGMENTS

The author wishes to express thanks to the Council of the Royal Free Hospital School of Medicine and to Dr. W. A. Leyshon for facilities for the completion of the work and to Mr. D. J. Morgan for his skilful preparation of the experimental plates.

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On the Pulse Method of Measuring Ultrasonic Absorption in Liquids

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Communicated by Sir Lawrence Bragg; *MS. received* 22*nd September* 1948

ABSTRACT. This paper deals with the experimental problems involved in accurate measurement of the absorption of ultrasonic waves in liquids. Reasons are given for preferring a method employing pulses of ultrasonic energy. The errors likely to be introduced by diffraction are discussed and it is shown that reliable measurements may be made in both the Fresnel and Fraunhofer regions. An account is given of a convenient method of correcting for divergence of the beam in the Fraunhofer region. The choice of the optimum conditions for accuracy is discussed and illustrated by practical examples. A description is given of the essential features of an apparatus working on six frequencies between 7·5 and 67·5 Mc/s. using the pulse technique.

§ 1. INTRODUCTION

The aim of this paper is to discuss some of the details involved in the measurement of the absorption of ultrasonic waves by the pulse method, which has already been described (Pellam and Galt 1946, Teeter 1946, Pinkerton 1947, Huntington, Emslie and Hughes 1948, Arenberg 1948). The discussion is illustrated by a short description of an experimental apparatus which has been used to obtain results described elsewhere (Pinkerton 1947, 1948). Measurements made in the Fraunhofer region of the quartz crystal require correction for the divergence of the beam (Grossmann 1932, Leonard 1940); a simple method of applying this correction is given. For the measurement of velocity the pulse method does not offer any substantial advantage over the acoustic interferometer or methods using diffraction of light, and the discussion will therefore be restricted to the problem of measuring the absorption alone.

All methods of measuring the absorption involve measurement of the intensity at a variable distance, x, from the transmitting crystal. The absorption coefficient α is then obtained from the relation $I=I_0e^{-2\alpha x}$. In the past the intensity has been measured by using the optical diffraction effects or the Rayleigh pressure of the sound. The inherent disadvantages of these two methods have been fully discussed by Fox and Rock (1941) and Willard (1941). Their conclusions will now be briefly summarized to show how the inherent inaccuracies of these methods may be avoided by using pulses of sound.

In order to detect the radiation more readily large powers have frequently been employed, as much as 100 watts being recorded (Sörensen 1936). Since all the sound is ultimately absorbed, this leads to an appreciable heating of the liquid. This heating in turn may alter the absorption coefficient, which varies quite rapidly with temperature in certain liquids. Local fluctuations in the temperature also cause refraction of the sound waves, which leads to further errors, especially if the path length is great. Moreover excessive intensities of sound cause cavitation, and the bubbles of gas released scatter and absorb the

sound, so that the absorption is found to depend on the amplitude, and on whether or not the liquid was previously degassed (Sörensen 1936, Fox and Rock 1941). The intensity required to produce cavitation in water is only about 1 watt/cm^2.

Unless the total absorption between the source and the detector is great, standing waves set up in the intervening space may alter considerably the variation in intensity with distance, and cause large errors in α.

The method based on the Rayleigh pressure has certain special disadvantages. The liquid cannot be stirred unless the pressure balance is specially protected, but without stirring absorption of even feeble intensities of sound inevitably raises the temperature of the liquid locally. The pressure balance must also be protected against bodily motion of the liquid, since the absorption of the wave by the liquid gives it momentum and therefore drives it away from the source.

In principle these difficulties of former methods can all be overcome in suitable ways. In practice the low intensity that must be used means that the sensitivity of the detecting system is too low to give precise results.

If measurements are made using pulses these difficulties can readily be overcome. Stirring the liquid has no ill effect, and heating is reduced by a large factor. The heating is proportional to the mean power radiated, which is the peak power multiplied by the "mark to space" ratio of the pulses. In the author's measurements the mean power was about 1/500 of the peak power. The peak intensity may be kept low enough to avoid cavitation; at the same time detection of much lower intensities is possible. A considerable advantage is that intensities can be measured conveniently and accurately by an electrical method, over a very wide range. The useful range of intensities is 100 to 1,000 times that of the pressure balance. This is because a tuned amplifier can be used to magnify the received pulses of sound, so that discrimination against random thermal movements of the liquid, i.e. ultrasonic "noise", is greater than with a pressure balance which responds to sound of all frequencies.

§ 2. DIFFRACTION EFFECTS

If sound is emitted from a circular crystal many wavelengths in diameter, one might assume, neglecting absorption, that the intensity is uniform everywhere inside a cylinder based on the crystal, and zero outside. It has been recognized that this approximation is only valid exceedingly close to the crystal, and off the central axis (Biquard 1935, Willard 1941, H. Born 1942).

A well known argument using Fresnel half-period zones shows that, along the axis of a circular crystal, the intensity oscillates between zero and a constant maximum value, so long as the axial distance, z, is less than z_1, where $z_1 \sim R^2/\lambda$; R is the radius of the crystal and λ the wavelength. Any point at a distance less than z_1 may be said to be in the Fresnel region, and any point at a greater distance, in the Fraunhofer region. Beyond z_1 the beam begins to diverge, and there are no more maxima and minima on the axis, until at sufficiently great distances the intensity falls off as $1/z^2$.

If measurements of the absorption of sound are to be exact these phenomena must be taken into account. In practice, for reasons which will become clear, the measurements must be made either entirely in the Fresnel region or entirely in the Fraunhofer region, and in the latter case the readings must be corrected to allow for the divergence of the beam. It will be convenient to consider separately the problems of making exact measurements in the two regions.

It is well known in optics that the Fresnel diffraction pattern of a circular aperture, in a plane normal to the axis, consists of a series of concentric circles. The number of circles increases with decreasing distance from the aperture. If the quartz crystal is assumed to be vibrating as a perfect piston, then its diffraction pattern will be the same as in the analogous optical case. Now suppose the absorption is measured in the Fresnel region of the crystal with a detecting device whose diameter is small compared with that of the crystal. It is then obvious that the measured intensity will not fall off uniformly with the distance, but will oscillate about the exponential diminution caused by true absorption.

If a perfectly plane detecting system which is bigger than the crystal is set perpendicular to the axis, then the measured intensity will involve a summation taken over the whole diffraction pattern. If the detector responds to the Rayleigh pressure it will be insensitive to phase and the appropriate summation is the arithmetic sum of the intensities at all points. For a crystal used as detector

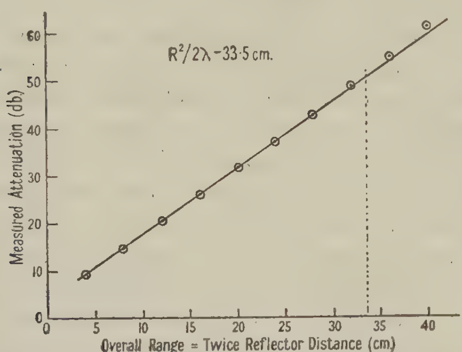

Figure 1. Measured attenuation plotted against distance to a large reflector in ethyl acetate.
Radius of the crystal $=0{\cdot}63$ cm.,
frequency $=21{\cdot}7$ Mc/s.,
temperature $=-6{\cdot}0°$ c.,
$R_2/2\lambda=33{\cdot}5$ cm.
The graph becomes non-linear beyond 33·5 cm.

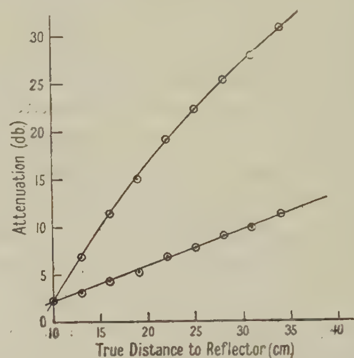

Figure 2. Corrected and uncorrected attenuation plotted against distance to a small reflector in the Fraunhofer region. Liquid: ethyl acetate.
frequency $=7{\cdot}47$ Mc/s.,
temperature $=-19{\cdot}7°$ c.,
$z_2=10$ cm.
Upper curve uncorrected, lower curve corrected points.

we must sum the amplitudes at all points vectorially. A mathematical treatment is evidently complex and in neither case has the problem been investigated. It is apparent however that proportional errors in α due to variations in the average intensity received by a crystal detector will be minimized if (*a*) the transmitting and receiving crystals are made as large as possible, or (*b*) the absorption coefficient per unit length is very large. For the errors caused by divergence to be negligible the total distance from transmitting to receiving crystal may not exceed R^2/λ. Without a mathematical treatment it is not certain that even this condition will be sufficient to avoid error. The limiting range can be decided by experiment however, and from the author's results it has been found that errors from divergence are negligible if z is not allowed to exceed $R^2/2\lambda$. At shorter ranges the plot of $\log I$ against z is perfectly linear, but if $z>R^2/2\lambda$ there is noticeable departure from linearity, as illustrated by the experimental results of Figure 1, for which $R^2/2\lambda=33{\cdot}5$ cm.

At ranges less than $R^2/2\lambda$ there is no systematic wandering of the points about the straight line. This fact shows that *the average* intensity received by a crystal,

whose diameter is the same as the sender, would be uniform within the Fresnel region in the absence of absorption. A common transmit–receive crystal, acting as receiver, is equivalent to a second crystal at twice the distance of an infinite plane reflector normal to the axis. We conclude that the slope of the straight line in Figure 1 accurately represents the absorption.

We now consider the case of measurements made in the Fraunhofer region where precise correction for divergence is essential. Fortunately an exact theoretical treatment for points on the axis has been given by Backhaus and Trendelenburg (1926), who showed that the intensity at distance z is proportional to $\sin^2[\frac{1}{2}k\{(R^2+z^2)^{\frac{1}{2}}-z\}]$, where $k=2\pi/\lambda$. This expression is also valid for axial points in the Fresnel region. The intensity at a point off the axis and well in the Fraunhofer region can be shown to be related to that on the axis by the expression

$$I_\beta/I_a=[\{J_1(x)\}/x]^2 \qquad \ldots\ldots(1)$$

where J_1 is a Bessel function of order one, $x=(2\pi R/\lambda)\sin\beta$ and β is the angle between the axis and a line through the point to the centre of the crystal.

To correct results of an experimental determination, which are observed directly in decibels, it is most convenient to calculate the correction also in decibels. $z_0=2R^2/\lambda$ is approximately the axial distance for which the aperture of the crystal is completely filled by a quarter-period Fresnel zone. Then if we express all the measured ranges in terms of the ratio $S=z/z_0$ we can show that the correction in decibels is determined only by S. A standard correction graph can therefore be plotted to find the correction if z_0 is known. Subtraction of the appropriate correction in decibels gives a set of readings of true absorption against distance. The measured intensity and therefore the corrected readings are relative to an arbitrary reference level, but since we are only interested in the slope of the graph of attenuation versus range this does not matter.

The calculation is made as follows.

Following H. Born (1942) we write the amplitude A_z in the form

$$A_z=A_0e^{-\alpha z}\sin[\tfrac{1}{2}k\{(z^2+R^2)^{\frac{1}{2}}-z\}] \qquad \ldots\ldots(2)$$

where A_0 is the amplitude at $z=0$. In practice $z^2 \gg R^2$, so we can write

$$A_z=A_0e^{-\alpha z}\sin(\pi R^2/2\lambda z). \qquad \ldots\ldots(3)$$

We have also

$$A_{z_0}=A_0e^{-\alpha z_0}\sin(\pi R^2/2\lambda z_0)=A_0e^{-\alpha z_0}\sin(\pi/4). \qquad \ldots\ldots(4)$$

Dividing (4) by (3) and substituting for z in terms of S, we find:

$$A_{z_0}/A_z=e^{-\alpha(z_0-z)}[\sin(\pi/4)/\sin(\pi/4S)]. \qquad \ldots\ldots(5)$$

If we now define A'_z and A'_{z_0} to denote the amplitudes of signals received by the transmitting crystal from a very small reflector placed at z_0 and z respectively, then we have

$$A'_{z_0}/A'_z=e^{-2\alpha(z_0-z)}[\sin(\pi/4)/\sin(\pi/4S)]^2. \qquad \ldots\ldots(6)$$

We may write equation (6) in the form

$$20\log(A'_{z_0}/A'_z)=40\alpha(z-z_0)\log e+40\log[\sin(\pi/4)/\sin(\pi/4S)]. \qquad \ldots\ldots(7)$$

The left hand side of equation (7) is simply the observed uncorrected attenuation in decibels below the arbitrary reference level A_{z_0}; let this be denoted D, which is a positive quantity for $z>z_0$. The first term on the right hand side represents

the true loss due to absorption, and the second the correction. Again for $z > z_0$ both terms are positive. Call the true attenuation D' and the correction C, then

$$D = D' + C \quad \text{or} \quad D' = D - C. \qquad \ldots\ldots(8)$$

Values of C for various values of S are given in Table 1.

Table 1

S	C (db.)	S	C (db.)	S	C (db.)	S	C (db.)	S	C (db.)
0·6	−5·40	2·5	14·36	4·5	24·40	6·5	30·76	8·5	35·36
0·8	−2·81	3·0	17·46	5·0	26·21	7·0	32·02	9·0	36·37
1·0	0	3·5	20·08	5·5	27·84	7·5	33·21	9·5	37·29
1·5	6·02	4·0	22·36	6·0	29·36	8·0	34·20	10·0	38·20
2·0	10·68								

If the corrected attenuations D' are plotted against z we should expect to find a straight line of slope

$$40(\log e)\alpha = 17{\cdot}35\alpha. \qquad \ldots\ldots(9)$$

To illustrate the procedure results of an experiment in ethyl acetate at −19·7° c. and 7·47 Mc/s. are given in Figure 2, in which values of both corrected and uncorrected attenuation are plotted. Although in this case the attenuation caused by divergence is 2·22 times that produced by true absorption, the corrected readings lie close to a straight line.

The plot of corrected attenuation against z has been found to be linear in many other experiments in which the values of z, z_0, R and the correction varied widely, and the agreement is considered to justify the making of the correction.

For measurements in the Fraunhofer region the diameter of the reflector is important. We have assumed above that the intensity is constant all over the face of the reflector, but if the reflector is too big this is no longer true. Since the intensity of the echo is proportional to the area, it is usually desirable to use the largest permissible reflector and under these circumstances we can calculate the maximum allowable diameter as follows.

Suppose an experimental accuracy of 1% is aimed at. If the total attenuation to be measured is 50 db., we must not allow an error of more than 0·5 db. in the amplitude of the signal returned from a reflector of finite size. The error will obviously be greatest when the reflector is at minimum range and least at maximum range. The reflector is assumed to be circular and placed normal to the beam symmetrically on the axis of the crystal. Then it can easily be shown that if the intensity at the circumference of the reflector falls 0·75 db. below that in the centre, then the average reflected intensity will fall by 0·5 db. But the variation in intensity with angular distance β off the axis is given by equation (1) from which we find that 0·75 db. corresponds to $x = (2\pi R/\lambda)\sin\beta = 0{\cdot}8$. If we assume that measurements will not be made at shorter range than z_0, we have $\sin\beta = R_{\max}/z_0$ where $R_{\max}$ is the largest permissible radius of the reflector. From these expressions it follows that $R_{\max} = (0{\cdot}8\lambda/2\pi R)z_0 = 0{\cdot}8R/\pi \simeq R/4$.

If the radius of the reflector is allowed to become too large the measured value of the absorption will be low. For measurements in the Fresnel region on the other hand, if the range of a large reflector is allowed to increase too far the

measured absorption will be too large, owing to divergence of the beam. A comparison of the results of measurements made in the same liquid at the same frequency and temperature in the Fresnel and Fraunhofer regions will therefore show whether both errors have been avoided.

§ 3. THE CHOICE OF THE BEST EXPERIMENTAL CONDITIONS

(i) *Optimum Range of Intensities*

Suppose α is found from the relation

$$I_1 = I_2 \exp\{-\alpha(z_1 - z_2)\} \qquad \ldots\ldots(10)$$

and the proportional error $\epsilon = \Delta I/I$ in I_1 and I_2 is constant, then the most probable error in α is given by

$$(\Delta\alpha/\alpha) = (\epsilon_1^2 + \epsilon_2^2)^{\frac{1}{2}}/\ln(I_2/I_1). \qquad \ldots\ldots(11)$$

We may make $\Delta\alpha/\alpha$ as small as we please without diminishing ϵ, by making I_2/I_1 sufficiently large, but we are limited by two conditions: (*a*) I_2 so great that cavitation occurs, (*b*) I_1 so small that it approaches the noise level of the detecting system.

If pulses are used the ratio I_2/I_1 may be made larger than in any alternative method of measurement. In the experiments of Pellam and Galt the ratio corresponded to about 120 db. In the apparatus described below the maximum value of I_2 was restricted by overloading in the first stages of the receiver. In this case I_2/I_1 corresponded to about 60 db.

(ii) *Common Transmission and Reception with a Single Crystal*

By the use of a common transmitting and receiving crystal the attenuation is doubled for a given length of liquid column, there is only one crystal to be mounted and matched electrically, and the mechanical design is simplified since the reflector alone need be movable. On the other hand the electrical circuits may be more complicated. The apparatus to be described was designed so that the frequency could be easily changed and the number of separate tuned circuits to be switched was reduced to a minimum by the use of a common T and R system.

(iii) *Choice of Fresnel or Fraunhofer Regions for Measurement*

Whether the measurements are carried out in the Fresnel or Fraunhofer regions is dictated by considerations of practical convenience. As we have already seen in § 2, if measurements are to be made in the Fresnel region the distance to the reflector must not exceed $R^2/4\lambda$. At low frequencies therefore, the Fresnel region may be so short that the total absorption in the region is too small to be measured accurately. In this case it is more convenient to work in the Fraunhofer region and to correct the observations for the divergence of the beam. The ratio of intensities I_2/I_1 may then be made as large as the apparatus will allow, even if α is quite small.

As a practical example consider water at 20° c. and let $\nu = 7{\cdot}5$ Mc/s., then $\alpha \simeq 0{\cdot}01\ \text{cm}^{-1}$ and $V = 1{\cdot}5 \times 10^5$ cm/sec. If $R = 1$ cm. the Fresnel region is limited to $z_c = R^2/2\lambda = 25$ cm. The corresponding attenuation is 2·17 db., which is clearly too small for accurate measurement. To increase the accuracy z_c must be increased to correspond to about 50 db., requiring the use of a crystal 4·8 cm.

in radius, which would be most inconvenient and expensive to make. These considerations may explain why some measurements of absorption made at very low frequencies have given inaccurate results. What was measured as absorption was largely caused by divergence of the beam.

In the above example the critical distance z_0 (cf. §2) is $2R^2/\lambda = 100$ cm. It may be inconvenient to work in the Fraunhofer region beyond 100 cm. from the crystal because of the amount of liquid required, and accordingly z_0 may be reduced by limiting the area of plating on the crystal or by restricting the aperture externally by means of an iris (§4).

(iv) *Accuracy of Setting the Reflector*

The criteria will be different in the case of measurements made in the Fresnel and Fraunhofer regions. In the Fresnel case the angular accuracy of setting is the only difficult criterion to meet. The error involved in treating the crystal and reflector as plane surfaces is assumed to be less than $\lambda/4$. In the case of a square crystal of side a with the reflector at very short range, the received echo falls to zero when the reflector is twisted from the setting for maximum signal through an angle $\Delta\theta \simeq \lambda/2a$. In the new setting the resultant signal received by one half of the crystal exactly cancels that received by the other. For a circular crystal $\Delta\theta$ will be modified by a numerical factor near to unity. In practice the signal must not fall by more than a small fraction of its amplitude, hence we have the condition $\Delta\theta \ll \lambda/2R$. For water at 75 Mc/s. $\lambda = 0{\cdot}002$ cm. and if $R = 0{\cdot}5$ cm. $\Delta\theta \leqslant 3{\cdot}6$ minutes of arc. Very accurate mechanical design is needed to preserve this accuracy of setting as the reflector is moved away from the source.

In the Fraunhofer case we have also to consider the transverse setting of the reflector in the beam. We have shown in §2 that a small reflector is necessary, and from the same considerations the transverse maladjustment permissible will be rather less than the radius of the reflector or roughly $\pm 0{\cdot}25$ mm. The fine mechanical adjustment required is described in §4 and the method of setting used in practice is described in §5. The additional adjustment of the angular position of the reflector need not be made as accurately as in the case of a large reflector in the Fresnel region, since the Fraunhofer diffraction pattern of the small reflector has a very broad central maximum.

(v) *Considerations in the Design of the Crystal*

The thickness of the crystal is fixed by the operating frequency but the radius may be chosen to suit the diffraction requirements. It is possible that the latter may conflict with electrical requirements. If the plated area of the crystal is made too small the effective parallel radiation resistance will be so large that it becomes difficult to match to the valve oscillator. Also a fixed external stray capacity will narrow the bandwidth more if the crystal is small.

The question of bandwidth has been discussed by Huntington, Emslie and Hughes (1948), who showed that for a crystal of acoustic impedance Z_0 in contact over one face only with a fluid of acoustic impedance Z_1, the effective Q of the crystal at resonance is given by

$$Q = (n\pi/4Z_1)[2(2Z_0^2 - Z_1^2)]^{\frac{1}{2}} \qquad \ldots\ldots(12)$$

where n is the order of the harmonic used to excite the crystal to resonance. For a quartz crystal with one face in water $Z_0 = 10Z_1$ so that, ignoring Z_1^2 in comparison with $2Z_0^2$, we find

$$Q = \frac{n\pi}{4}\frac{2Z_0}{Z_1} = 5n\pi. \qquad \ldots\ldots(13)$$

On the fundamental the Q is therefore about 16.

The width of the resonance curve is such that $Q = f/\Delta f$, where Δf is the difference between the frequencies exciting the crystal to half the maximum power. It can be seen from (13) that Δf is numerically the same on all the harmonics, although Q itself is proportional to n. This means that if the bandwidth is adequate to transmit pulses of a given width on the fundamental it will also be adequate on the harmonics. A Q of 16 will allow the transmission of pulses of duration about 1 microsecond at 15 Mc/s., or about 2 microseconds at 7·5 Mc/s. To obtain a pulse with a sensibly flat top, however, its duration must exceed the minimum value by a factor of 2 or 3. If the absorption is very high difficulties may be caused because the reflector must be placed extremely close to the crystal, with the result that the transmitted pulse and its echo become merged together; moreover under these conditions the effects of paralysis in the receiver are most severe. The bandwidth of the crystal has been sufficient to allow measurement of absorption up to 30 db/cm. To measure absorption higher than this one may use a quartz bar as a reflector which also delays the echo (Rapuano 1947); this technique makes it unnecessary to generate very short pulses. No attempt has therefore been made to generate pulses narrower than 1 microsecond.

(vi) *Effect of Pulses on Experimental Accuracy*

If α varies with ν there must be some error involved in measuring with short pulses, and Pellam and Galt (1946) showed that when α varies as ν^2 the error is about 1 part in 200 for one-microsecond pulses at 15 Mc/s.

(vii) *Control of Temperature*

The importance of maintaining a uniform temperature in the liquid by stirring cannot be overstressed. This requires a more uniform temperature than is needed to remove errors caused by true variations in α with T, because local fluctuations cause refraction of the beam and "wandering" of the size of the echo. Experience shows that errors are likely if the temperature is not maintained uniform to $\pm 1/10°$ c. throughout the liquid.

§ 4. EXPERIMENTAL EQUIPMENT

The requirements to be fulfilled by the apparatus were the following. An experimental accuracy of 1% was aimed at. Measurements were to be made on as many frequencies as possible and easy changing of the frequency was essential. Means had to be provided for varying, controlling and measuring the temperature to about $\pm 0·1°$ c. Measurements were to be made in water or in any organic liquid.

The preliminary measurements were made at 15 Mc/s. in water. On this frequency a large volume of liquid was required as the absorption is low, and in view of this it was decided to extend the measurements to higher frequencies.

Use of the odd harmonics of the 15 Mc/s. crystal would leave a rather large gap between the fundamental and the third harmonic. A crystal cut for 7·5 Mc/s. and worked on odd harmonics up to the 9th would give a more even distribution of frequencies if 15 Mc/s. also were included in the series. It was decided to construct a multi-channel transmitter-receiver using switching to change frequency. A superheterodyne receiver was used, so that a common I.F. amplifier could be used on all frequencies. For the reasons already discussed a common transmit and receive system was used. It was decided to include time calibration marks and a strobe time-base, in case it was desired to measure velocity.

A schematic diagram of the apparatus is shown in Figure 3. It is identical in principle with that of Pellam and Galt (1946). The main difference is that

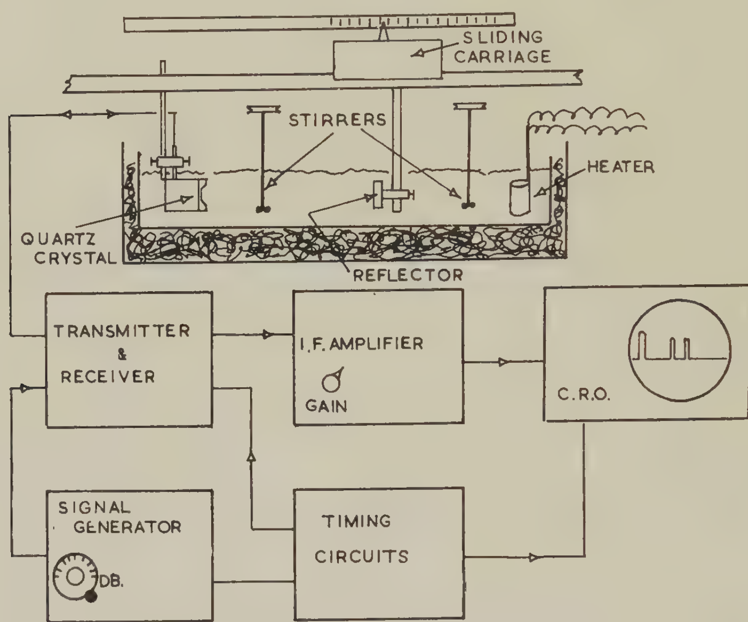

Figure 3. Schematic diagram of the apparatus.

in this case the output from a signal generator is passed through a calibrated attenuator before injection into the amplifier to provide a standard for comparison. The amplitude of an echo is compared on an oscilloscope with that of a pulse from the signal generator injected into the quartz crystal itself. Variations in the gain of the amplifier affect both signals equally and consequently introduce no error in the measurement.

We now consider the design of the more important items in detail.

(a) *Mounting the Crystal*

If the assumptions made in the discussion of § 2 are to be justified the crystal must be held quite flat in its mount, and must not be distorted by stirring of the liquid. A mounting which supports the crystal from behind is essential; an unsupported crystal was found to be distorted seriously if the liquid was stirred. The mounting adopted was kindly suggested to the author by Mr. T. Gold.

The main part of the mount is a cylindrical brass block 2 in. in diameter, bored out to take a $\frac{3}{4}$ in. diameter dystrene plug, which in turn has a central brass plug of $\frac{1}{2}$ in. diameter. These parts are slightly tapered and forced into one another. The end of the cylinder is ground flat and polished and the crystal is held against it by a clamping ring and rubber washer. The area of crystal excited is determined by the brass plug which forms the live electrode. The use of rubber washers prevents measurement in liquids such as benzene which diffuse into rubber. A mounting is now under development in which the crystal is soldered into position.

The crystals used are X-cut, 1 in. in diameter and gold-plated on the face in contact with the liquid. The plating is continued round the edge to cover a ring on the reverse side extending inwards for 1/16 in. This ring serves to earth the plating on the front. The provision of the $\frac{1}{4}$ in. unexcited border to the crystal has been found to reduce a tendency to spurious modes of oscillation resulting in false echoes immediately following the transmitter pulse.

(*b*) *Mechanical Adjustments*

From § 3 we see that it is necessary to have two angular adjustments to the position of the crystal mounting and two angular and two linear adjustments to the position of the reflector. The angular adjustments are made by spring-loaded screws at the ends of short levers, and geometrical slides provide the linear adjustments. All the adjusting mechanisms for the reflector are mounted on a sliding carriage running on two 1 in. steel bars, to which the adjusting mechanism for the crystal is clamped. The bars are supported above the trough containing the liquid by four pillars screwed on to a 6 in. channel iron baseplate. The whole assembly has the required exceptional rigidity and is mounted on rubber to prevent vibrations of the reflector.

(*c*) *Irises and Reflectors*

We have already said that it may be desirable to restrict the effective diameter of the crystal by an iris. If multiple reflections of the pulses occur between the crystal and the iris the numerous echoes visible on the oscilloscope obscure the wanted signal. This difficulty may be avoided by making the iris in the form of a truncated cone, with a hole bored through the narrower end, so that sound striking the cone is reflected sideways and does not return to the crystal. A cone of 45° semi-angle should be avoided otherwise the radiation can return along its own path if reflected from the sides of the tank. The cones are made of brass and rigidly supported with the aperture a few millimetres from the surface of the crystal. The aperture is chosen to make z_0 the desired value (cf. § 3).

The reflectors are of two kinds. The large plane reflector, for use in the Fresnel region, is made of brass $\frac{1}{4}$ in. thick and 1 in. square. The surface is ground and polished. The accuracy required in the surface is $\lambda/4$ at 70 Mc/s., which is approximately 1/1000 in.

Small reflectors to be used in the Fraunhofer region must project forward about one inch from the supporting rod, so that the latter gives an independent reflection which can be ignored. The tip of the reflector should be plane and slightly larger than the supporting shank. Brass reflectors of 1 mm. and 2 mm. diameter have been used successfully at 7·5 and 15 Mc/s.

(*d*) *Transmitter-Receiver Unit*

The circuit of the transmitter-receiver unit has certain unusual features. The aim of the design is to reduce the number of tuned circuits required, and the arrangement adopted uses three on each of five frequencies involving fifteen preset adjustments to the tuning. A simplified circuit diagram is given in Figures 4(*a*) and 4(*b*). The frequency of oscillation is determined by the grid-cathode circuit of the oscillator valve, which is a 6AG7 pentode. The coil in the anode of the oscillator acts as tuned circuit both for the transmitter output and for the receiver input and a diode mixer is connected directly across this circuit. The local oscillator E.M.F. and an R.F. pulse from the signal generator are also fed into this tuned circuit. The oscillator valve is switched on by a positive pulse on its screen. To prevent oscillations from continuing after the end of the pulse the screen normally has a negative bias of about 20 volts generated in the cathode circuit.

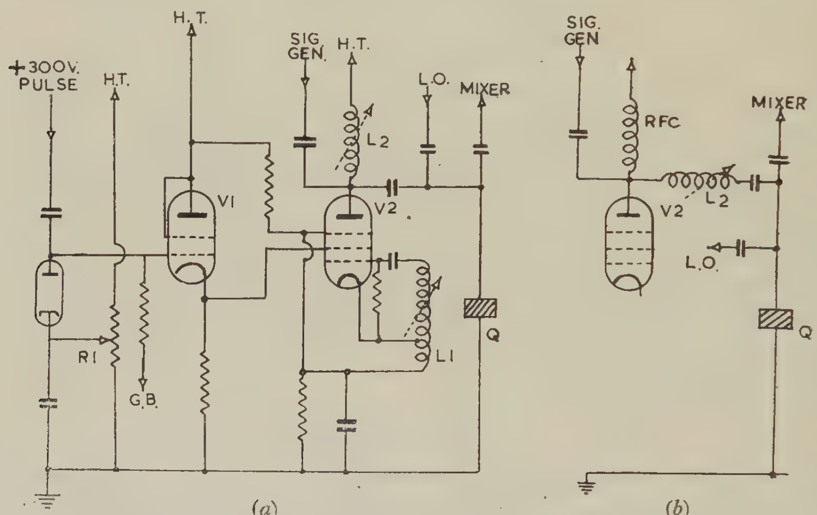

Figure 4. (*a*) Circuit of transmitter and quartz crystal at 7·5 and 22·5 Mc/s. (*b*) The same rearranged for the frequencies 37·5, 52·5 and 67·5 Mc/s.

At high frequencies the total inductance required in the anode circuit becomes very small; the inductance of the coil itself may then be so small in comparison with that of the leads that tuning is impossible. On the two lowest frequencies all the stray capacities appear in parallel, as is evident from the circuit of Figure 4(*a*); on the higher frequencies, however, the total stray capacity is divided into two parts appearing in series with one another across the coil by the rearrangement shown in Figure 4(*b*). In this way a coil may be used which is large enough compared with the lead inductance to be easily tuned.

Owing to the necessity for changing frequency it is not possible here to match both ends into the screened lead connecting the crystal to the oscillator. A switched matching circuit in the crystal housing itself, which is below the surface of the liquid, was considered impracticable. It was therefore essential that this lead be kept as short as possible. The transmitter and receiver circuits, with the local oscillator and first I.F. amplifier stages, were therefore designed as one small unit which could be mounted just above the troughs containing the liquid.

The power output from the quartz crystal is controllable by varying the size of modulation pulse fed to the screen of the 6AG7. The bias on the diode can be set to limit the pulse at the grid of V_1; thus the setting of the potentiometer R_1 controls the power output from the oscillator.

(*e*) *Other Electrical Circuits*

The other electrical circuits follow normal radar practice; it will therefore be sufficient to state the types of circuits used. The whole equipment runs at a recurrence frequency of 250 c/s. determined by a crystal calibrator taken from a Gee Mk II equipment (Dippy 1946). The pulse modulating the transmitter is generated by a multivibrator (Williams 1946) and is variable in length from 2 to 40 microseconds. The pulse modulating the signal generator is variable in width and is delayed a variable amount by a Phantastron circuit (Williams and Moody 1946). The main I.F. amplifier on 15 Mc/s. is taken from an H_2S Mk II radar set (Carter 1946) and is modified to include a cathode follower feeding the output to the oscilloscope and a neon stabilizer to control the voltage on the screens. The Sanatron time base (Williams and Moody 1946) has three ranges of 80, 400 and 2,000 microseconds duration. A very stable delay using the Sanatron was made to allow a variable delay of from about 300 to 2,000 microseconds in the start of the time-base. This feature allows the time-base to be used as a strobe if accurate measurements of velocity are made. The signal generator is a commercial model fitted with a piston attenuator; it covered all the frequencies used in the measurements.

(*f*) *Temperature-controlled Bath and Thermostat*

Several troughs have been employed depending on the liquid concerned. For water a double walled trough 5 ft. in length was made and water could be circulated in an outer jacket from a thermostat. For liquids having a higher absorption than water, heating in a small lagged trough with an electric heater controlled by a Variac proved satisfactory. The temperature could be controlled manually to a satisfactory degree of accuracy. In the trough run two high speed stirrers of $\frac{1}{2}$ in. diameter. The speed is controlled so that bubbles which might get in the beam and alter the amplitude of the echo are not sucked down into the liquid. Temperature is measured with ordinary mercury thermometers calibrated in tenths of a degree centigrade. At temperatures below 0° C. a pentane thermometer is used. To cool the liquid below room temperature solid carbon dioxide and liquid air are employed. The refrigerant is placed in a small can immersed in the liquid. Whether or not the temperature is sufficiently uniform for accurate readings to be taken can be seen immediately from the oscilloscope. If the echo is varying constantly in size the temperature is not uniform throughout the liquid.

§5. METHOD OF CARRYING OUT AN EXPERIMENT

(i) *Setting up the Apparatus*

The setting up falls naturally into two parts, concerned with the electrical circuits and mechanical adjustments respectively.

The width of the transmitted pulse was made just wide enough to give a flat top to the echo seen on the oscilloscope. The power output was adjusted so that when the reflector was at minimum range the signals could be accepted by the receiver without overloading. This could be detected by setting the

pulse from the signal generator at various points on the time-base and noting whether it suffered any alteration in amplitude. To facilitate comparison the pulse from the signal generator was always placed close to the received echo and adjusted to be of the same width. The frequency ν of the oscillator valve (Figure 4) was set to give the maximum size of echo and the frequency was afterwards measured. The optimum frequency was found to vary with the temperature of the liquid, owing to changes in its acoustic impedance and to thermal expansion of the mounting of the crystal. The frequency was therefore reset after each change in the temperature. Due allowance was made for this change in ν when calculating α/ν^2 from the results. Operation too far from the resonant frequency caused distortion of the pulse. The frequency was measured with a crystal check wavemeter which had an accuracy for c.w. signals of better than one part in 1,000. The pulse width was always set to its maximum value when measuring the frequency so that the "spread" on the wavemeter was a minimum.

As has been stated measurements are made with the reflector in both the Fresnel and the Fraunhofer diffraction regions; the mechanical adjustments required to put the crystal housing and the reflector in their correct positions differed in the two cases.

For measurements in the Fresnel region the transverse adjustments of the reflector perpendicular to the beam are not critical, since the reflector is much bigger than the source. The adjustment of the crystal mounting so that the beam travels exactly parallel to the guide rails may be made by eye. The angular adjustment of the reflector to be perpendicular to the beam is extremely critical however and must be performed with care by turning the two adjusting screws alternately. The correct adjustment may be checked in the following manner: if only one screw is turned the echo should go through one large maximum value, with smaller subsidiary maxima on either side. The occurrence of two or more maxima of nearly equal size indicates that the original setting was incorrect.

In the Fraunhofer region, as we have shown in § 3, the adjustments are more critical. It was found convenient to make the adjustments in the following sequence. The reflector is first placed at the nearest distance at which it is intended to observe. It is then moved in the two directions perpendicular to the beam until the echo reaches a maximum. Adjustment of the angle of the reflector follows, and then a further transverse adjustment. It is now necessary to ensure that the beam is travelling parallel to the guides on which the reflector is moved. This is done by placing the reflector at maximum range, and making angular adjustments to the crystal housing. On replacing the reflector at minimum range the reflector adjusting screws will probably require resetting. The process of alternate adjustment of the reflector at short range and the crystal housing at long range is carried on until no further improvement in signal strength can be obtained in either position. It is important that the last adjustment to be made should be at the nearer range since the two transverse adjustments of the reflector are by far the most critical, due to the narrowness of the central maximum of the beam at short ranges.

(ii) *Procedure for Measurement*

The temperature is brought to a steady value and measured. A series of readings of the strength of the echo at increasing ranges is taken by comparing it with a pulse on the same frequency from the signal generator. The amplitudes

are measured directly in decibels of attenuation. If correction is required for divergence the absolute value of the range must be recorded; if no correction is required increments of range only need be noted. Usually between ten and fifteen readings are taken at equal range intervals and plotted on a graph (cf. Figures 1 and 2).

If for any reason the temperature changes slowly throughout an experiment, the error can be minimized by taking readings of attenuation with range increasing and then at the same points with range decreasing. Taking the average of the two readings for each point removes the error; the corresponding temperature is also taken as the average of the values before and after the experiment. This method has been used at low temperatures, where thermostatic control is inconvenient.

ACKNOWLEDGMENT

The author wishes to thank Mr. J. A. Ratcliffe for many discussions on the diffraction phenomena of a radiating quartz crystal and for much helpful advice in the preparation of the paper. He also wishes to thank Mr. J. Fuller for helping to design and for making the mechanical components of the apparatus.

The work was carried out during the tenure of a research scholarship at Trinity College and a maintenance grant from the Department of Scientific and Industrial Research; this provided funds to pay for some of the equipment.

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The Confinement of Slow Charged Particles to a Toroidal Tube

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Communicated by R. E. Peierls ; MS. received 18th October 1948, and in amended form 9th November 1948

ABSTRACT. In the magnetic field of an infinite straight filament current a slow charged particle is confined near the current but drifts parallel to it. This suggests the field of a circular loop of current as a means of confining a particle within a toroidal tube, and it is found that a particle is so confined in such a field if its velocity is sufficiently small in relation to the field. The results obtained are used to examine briefly the possibility of a magnetic self-constricting effect in electric sparks and arcs.

§ 1. INTRODUCTION

It is well known that the motion of a slow (non-relativistic) charged particle in a uniform magnetic field is confined to the vicinity of a particular line of force. This might lead one to think that in the magnetic field of a straight current filament, in which the lines of force are circles, the particle would still tend to remain near one particular line of force, and this would make it possible to confine such particles to the inside of a toroidal tube. It will be shown that this conjecture is incorrect, owing to the fact that the motion includes a drift parallel to the current.* This result logically suggests the magnetic field of a circular loop of current as a means of confining the particles. The present paper is mainly devoted to a discussion of this case, and it will be shown that, provided the particle velocity is sufficiently small in relation to the field, all orbits are confined to the vicinity of the current loop. If the current producing the magnetic field is itself due to the charged particles considered, one may ask in what circumstances such a filament can be self-confining. This question is considered in § 5.

The treatment in the present paper is idealized in so far as it ignores the effects of collisions and stray fields.

§ 2. MOTION IN THE FIELD OF A STRAIGHT CURRENT FILAMENT

If a current I (E.M.U.) flows in a direction which is taken as the positive direction of the z axis, using cylindrical coordinates r, θ, z, the magnetic field components are

$$H_r = H_z = 0; \qquad H_\theta = 2I/r. \qquad \ldots\ldots(1)$$

The equations of motion of a particle of charge e (E.S.U.) and mass m grammes in this field are

$$r^2\dot{\theta} = \text{a constant, } p \text{, say,} \qquad \ldots\ldots(2)$$

$$\ddot{r} = h(K - h\ln r)/r + (p^2/r^3), \qquad \ldots\ldots(3)$$

$$\dot{z} = K - h\ln r. \qquad \ldots\ldots(4)$$

In these equations h has been written for $2eI/mc$ (c being the ratio of the electromagnetic to the electrostatic unit of charge) and K is an integration constant

* This result is probably not new, but does not appear to have been published ; the present paper therefore includes a section on this case.

defined by $K=\dot{z}_0+h\ln r_0$, where $\dot{z}_0$, r_0 are the values of $\dot{z}$, r at some arbitrary point on the orbit. Equation (2) represents the conservation of angular momentum about the z-axis.

Since equation (3) contains no $\dot{r}$ term, the radial force acting on the particle is a function of its radial position alone, and we may therefore imagine $\ddot{r}$ to be derived from some scalar potential $V(r)$ according to the relation

$$\ddot{r}=-\frac{dV(r)}{dr}=\frac{h}{r}(K-h\ln r)+\frac{p^2}{r^3}. \qquad \ldots\ldots(5)$$

Equation (5) may be integrated once to give V expressly as a function of r which may be shown to have always the general form given in Figure 1, i.e. V is continuous

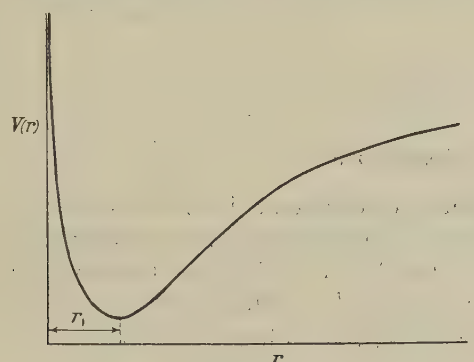

Figure 1.

throughout the range $0<r<\infty$, becoming infinite at $r=0$ and $r=\infty$, and having only one minimum in between. Clearly such a form for the potential function ensures that the radial part of the motion is stable, with r varying in some periodic manner about the value r_1 at which V is a minimum. The singularity at $r=0$ shows that a particle with finite kinetic energy never actually reaches the current filament.

To examine the stability in the z-direction we must use equation (4), which gives for the z-displacement after time T:

$$z_T=\int_0^T (K-h\ln r)\,dt. \qquad \ldots\ldots(6)$$

We have already seen that in general the particle oscillates radially about the value r_1. The special case of a non-oscillating particle with this value of r may be realized by suitable initial projection, and for such a particle (6) becomes

$$z_T=(K-h\ln r_1)T. \qquad \ldots\ldots(7)$$

In the equation for K, r_0 is arbitrary and may therefore be chosen equal to r_1. Substituting this value for K in (7) we have

$$z_T=\dot{z}_0 T, \qquad \ldots\ldots(8)$$

which shows that there is a steady drift parallel to the z-axis. By using an expansion about this solution the z-motion of an oscillating particle can be investigated, and z_T shown to contain (in addition to periodic terms) terms representing a drift. A more detailed examination of the actual orbit shows that the drift is always in the same direction, however the particle be projected initially. If we suppose the current producing the field to consist of a stream of negative charges, then a

negatively charged particle drifts in the opposite direction to this stream, a positively charged particle in the same direction. Only in the trivial case of vanishing projection velocity does the drift vanish.

§ 3. MOTION IN THE FIELD OF A CIRCULAR LOOP OF CURRENT

Let us now suppose that the straight filament current of the last section is bent round to form a closed circular loop. In this case a drift which follows the current merely takes the particle repeatedly round the loop and does not enable it to escape. However, the magnetic field is now more complicated, having two directions of curvature at any one point, and this may conceivably affect the stability of motion in some other component, viz. the r- or z-component in the cylindrical coordinates in which the origin is the centre of the loop and the z-axis the axis of the loop. The present section investigates this stability.

In cylindrical coordinates the Lagrangian of the particle is given by

$$L=\tfrac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2)+(e/c)(r\dot{\theta}A_\theta+\dot{r}A_r+\dot{z}A_z), \qquad \ldots\ldots(9)$$

where $\mathbf{A}$ is the magnetic vector potential defined by $\mathbf{H}=\operatorname{curl}\mathbf{A}$. The symmetry of the problem is such that A_θ is the only non-vanishing component of $\mathbf{A}$, and we may therefore write A for A_θ. It is also clear from symmetry that A is independent of θ, and hence from (9) L is independent of θ. This being so, Lagrange's equation of motion in the θ, $\dot{\theta}$ coordinates becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=0, \qquad \ldots\ldots(10)$$

so that

$$\partial L/\partial\dot{\theta}=\text{a constant, } P, \text{ say.} \qquad \ldots\ldots(11)$$

Differentiating (9) with respect to $\dot{\theta}$ and substituting in (11) gives

$$mrv_\theta+erA/c=P, \qquad \ldots\ldots(12)$$

where v_θ has been written for $r\dot{\theta}$, the azimuthal component of the linear velocity. Since the magnitude of v_θ never exceeds that of the resultant velocity v, which in a magnetic field is constant in time, we have the important inequalities

$$c\,(P-mrv)/e\leqslant rA\leqslant c\,(P+mrv)/e, \qquad \ldots\ldots(13)$$

which can be used directly to investigate the stability of the motion.

In order to see whether the restriction which (13) places on the value of rA limits the spatial extent of the orbit, we examine the curves of rA against r, with z as a parameter. From symmetry, equal and opposite values of z give the same (rA, r) curve. We shall need the following well-known expressions for the magnetic vector potential and field components of a filament current I flowing in a circular loop of radius a:

$$A=4aI\int_0^{\pi/2}\frac{(2\sin^2\theta-1)\,d\theta}{\{(a+r^2)+z^2-4ar\sin^2\theta\}^{\frac{1}{2}}}, \qquad \ldots\ldots(14)$$

$$H_r=-\frac{\partial A}{\partial z}, \qquad \ldots\ldots(15)$$

$$H_z=\frac{1}{r}\frac{\partial}{\partial r}(rA). \qquad \ldots\ldots(16)$$

Evidently both A and H_z are finite on the z-axis, hence both rA and $\partial(rA)/\partial r$ vanish at $r=0$. At large distances the field is that of a magnetic dipole, so that rA and $\partial(rA)/\partial r$ also vanish at $r=\infty$. We know further that H_z, and hence rH_z, changes sign once as r increases from zero to infinity. Assuming the current direction in the loop to be such that the change is from positive to negative H_z with r increasing, we see from (16) that $H_z=0$ corresponds to a maximum in rA. The smaller z, the higher this maximum, and the nearer it is to $r=a$. In the $z=0$ plane at $r=a$, A has a $1/(a-r)$ singularity and H_z a $1/(a-r)^2$ singularity. The general form of the (H_z, r) curves and the (rA, r) curves with z as a parameter can now be sketched. They are shown respectively in Figures 2 and 3. Since H_r is zero both for r equal to zero and for r equal to infinity, (15) confirms that all the $(rA–r)$ curves coincide at the origin and at infinity.

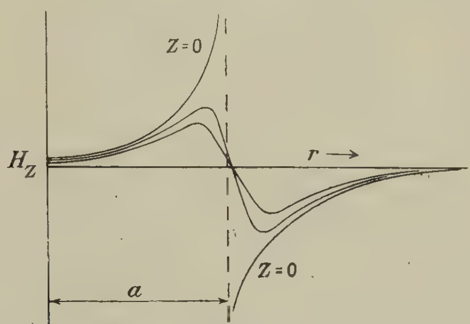

Figure 2.

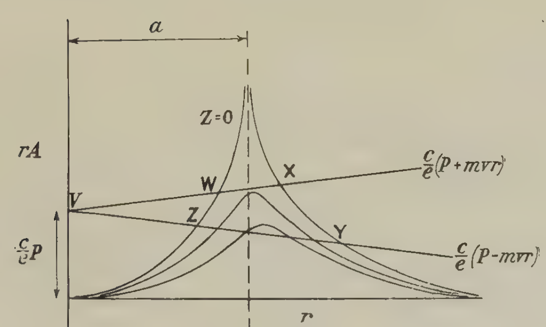

Figure 3.

In Figure 3 have been drawn also the pair of straight lines $rA=c(P\pm mvr)/e$, intersecting the $z=0$ curve in the four points W, X, Y, Z. However the particle moves, rA must, according to (13), always lie between these lines and the two branches of the $z=0$ curve. Thus the r and z coordinates of the particle are constrained to values in the area WXYZ, and provided these four intersections exist the particle does not escape but moves always within some finite region around the current filament. This region has the form of the space between two toroidal tubes, the outer enclosing the inner, and each enclosing the current filament. It is clear from Figure 3 that $rA=c(P+mvr)/e$ determines the inner boundary surface and $rA=c(P-mvr)/e$ the outer. It should be noted that although these boundaries enclose all possible orbits having a definite P and v, any particular orbit will, in general, occupy a smaller region.

From Figure 3 it is clear that the intersections W, X always exist provided $OV = cP/e$ is positive. That is to say, a closed inner boundary exists and the particle can never reach the current filament when cP/e is positive. When cP/e is negative it is not so easy to see from Figure 3 what happens to the inner boundary, but (12) shows that it must exist in this case also, for if it did not rA would tend to $+\infty$ as the particle approached indefinitely close to the current filament, and with mrv_θ continuing finite P would have to be positive, which is contrary to hypothesis. The inner boundary then always exists: no orbit intersecting the current filament is permitted to the particle. In the previous section this was already seen to be true of a particle in the field of an infinite, straight current filament, which may be regarded as a special case of a circular current loop of infinite radius.

A closed outer boundary exists only if the intersections Y and Z exist. It is obvious from Figure 3 that the necessary and sufficient condition for the existence of Z is that cP/e be positive (in the limit when cP/e is zero Y and Z coincide at the origin). This, however, is not sufficient to ensure the existence of Y: for a given value of P, in order that Y shall exist the slope of the line $rA = c(P - mvr)/e$ must not be less than the slope of the tangent from V to the $z=0$ curve. This limiting slope is a function of P and is negative, say $-s(P)$. We have then as the criteria for the existence of a closed outer boundary, first that cP/e must be positive, and second that for any particular positive cP/e, v must not exceed $es(P)/mc$. If the outer boundary is closed the motion is certainly confined; if it is open the motion may or may not be confined, a calculation of the orbit being in general necessary to decide this. There is, however, one case in which the motion is obviously unstable, namely that of a particle moving along the z-axis. Such a particle has no interaction with H_z, and since H_r vanishes on the axis (by symmetry), the particle drifts uniformly along the z axis.

§ 4. APPROXIMATE SHAPE OF LIMITING SURFACES NEAR THE CURRENT FILAMENT

In this section it will be assumed that the particle is not merely confined by the field of the circular current loop, but is confined so severely that its greatest distance from the current never exceeds a certain small fraction of the radius a of the loop. Consideration of (12) in conjunction with Figure 3 shows that this can be achieved by projecting the particle in such a manner that A is large compared to mcv/e. With this assumption we proceed to investigate the approximate shape of the limiting surfaces.

We already know that the boundaries are surfaces of revolution about the z-axis whose generating sections by any rz plane have the equations

$$c(P \pm mvr)/e = rA(r, z), \qquad \ldots\ldots(17)$$

the positive sign on the left-hand side being taken for the inner and the negative for the outer boundary. The approximate form of these equations when $|r-a|$ and $|z|$ are small compared to a is found by putting expression (14) for A into a form involving complete elliptic integrals and using the standard series expansions for such integrals, carried to the necessary number of terms in $(r-a)/a$ and z/a. Neglecting the second and higher powers of such terms, equations (17) become, after rearrangement, and with x written for $(r-a)$,

$$\ln(x^2+z^2) = -4 + 2\ln 8a - \frac{c}{Iea}\left\{(P \pm mva) + \frac{x}{2a}(-P \pm mva)\right\} \qquad \ldots\ldots(18)$$

or $$x^2+z^2=B\exp(Dx), \qquad \ldots\ldots(19)$$
with $$B\equiv 64a^2\exp\{-4-c(P\pm mva)/Iea\} \qquad \ldots\ldots(20)$$
and $$D\equiv c(P\mp mva)/2Iea^2. \qquad \ldots\ldots(21)$$

Introducing polar coordinates about the current filament as origin by means of the transformations $x=\rho\cos\phi$, $z=\rho\sin\phi$, (19) becomes

$$\rho^2=B\exp(D\rho\cos\phi). \qquad \ldots\ldots(22)$$

Investigation of this curve shows that ρ is a maximum at $\cos\phi=+1$ and a minimum at $\cos\phi=-1$. Relative values of ρ_{max} and ρ_{min} for the outer and inner boundaries are given in the Table.

Outer Boundary		Inner Boundary	
(1)	(2)	(3)	(4)
ρ_{max}/a	ρ_{min}/a	ρ_{max}/a	ρ_{min}/a
0·100	0·078	0·053	0·046
0·0100	0·0096	0·0034	0·0033

The figures in columns (2), (3) and (4) were calculated from (18) by assuming the values in column (1) and taking mva/ρ as 0·1. (Provided the latter ratio is small, which follows automatically from the assumption $A \gg mcv/e$ used in deriving (18), the results are not very sensitive to its actual value. Thus, if we assume it to be 0·01 instead of 0·1, the first entry in column (2) becomes 0·080, and the second is still 0·0096 to the accuracy of the Table.) The ratio ρ_{max}/ρ_{min} for each boundary surface is a measure of the extent to which its cross-section by an rz plane differs from a circle.

§ 5. SELF-CONFINING EFFECT OF A CURRENT

Either type of magnetic field, cylindrical or toroidal, so far considered has been assumed produced by an electric current of infinitesimal cross-section. We now consider the case of an electric arc or spark, which consists of a stream, of infinite cross-section, of similarly charged particles.* A magnetic field, due to the current, will exist both inside and outside such a stream. The internal field will have an effect on the motion of the particles carrying the current, and this effect is qualitatively discussed in the present section.

We have seen that in both a cylindrical and a toroidal magnetic field a charged particle drifts parallel to the current producing the field, while remaining (for sufficiently small velocities) always in the vicinity of the current filament. It is therefore reasonable to suppose, when we are considering a current of finite cross-section, that whether the current be cylindrical or toroidal in form the effects of the internal magnetic field on the particles carrying the current will be qualitatively similar.

To fix our ideas we suppose the particles constituting the current to be distributed uniformly in a cylinder of infinite length and radius b, all the particles having initially the same velocity parallel to the axis of the cylinder. Outside this cylinder the magnetic field is the same as if all the current were concentrated on its axis, i.e. it is still $2I/r$ in the notation of §2. Inside, the field increases uniformly from zero on the axis to $2I/b$ at the surface. This internal field acting

* Electrons moving through a relatively stationary distribution of positive ions.

on the charged particles producing the current tends to keep them radially confined and to give them an axial drift which reinforces their original drift. (The direction of the axial drift imparted by the field depends on the sign of the field gradient; in §2 we saw that the external $(2I/r)$ field imparted a drift opposite to that of the current-carrying particles.) A self-concentrating effect of the particles in the magnetic field of their own drift current would then appear possible. Such a tendency must, however, be balanced against certain deconcentrating effects such as the thermal agitation and mutual electrostatic repulsion of the particles.

On the basis of the present simple model an estimate was made of the minimum current necessary for a self-concentrating effect to be possible. The limit was found to be set by the inequality $I^2 > NT$, where N is the number of particles per centimetre length of current and T the mean kinetic energy (in ergs) per particle. Using typical values of I and N for the electric spark in air (cf. Allibone and Meek 1938), it was estimated that a self-concentrating effect would be possible only if T were so low as to correspond to an electron temperature of less than 10° K. This is so far from actual electron temperature in sparks that, even allowing for the crudeness of our estimate resulting from an over-simplified model, one would hardly expect the constrictive effect of its own magnetic field to be appreciable in the atmospheric spark. On the other hand, in electric arcs, carrying large currents, such an effect is possible. The detailed theory of this so-called "pinch effect" in arcs has been given by Tonks (1939).

ACKNOWLEDGMENTS

The author is indebted to Professor R. E. Peierls for his suggestion of this problem and general guidance of the work, and to Mr. T. H. R. Skyrme for a number of helpful discussions.

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Tonks, L., 1939, *Phys. Rev.*, **56**, 360.

Observations made on the Ionosphere during Operations in Spitsbergen in 1942-43

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Communicated by R. L. Smith-Rosé; MS. received 3rd August 1948, in revised form 20th October 1948

ABSTRACT. Observations on the ionosphere were made in Spitsbergen (latitude 78° N., longitude 15° E.) in 1942–43 on behalf of the Admiralty. Observations in such a high latitude have seldon been made. The salient features of the equipment and site are described, the equipment proving excellent except for the magnetograph.

Each region of the ionosphere is then considered in turn; the same main regions are found in Spitsbergen as elsewhere, but there are many abnormalities, of which the "Polar Spur" is perhaps the most interesting. Ionization often changes with great rapidity. The effects of magnetic storms are also described

§ 1. INTRODUCTION

MEASUREMENTS on the ionosphere were made at Barentsburg, Spitsbergen, latitude 78° North, longitude 15° East, from 12th October 1942 to 8th June 1943. After 24th October, records of the height and critical frequency were obtained hourly with few exceptions, and, in addition, a number of records spaced at three-minute intervals and 15-second intervals were obtained from time to time. Continuous observation of the variation of the vertical component of the earth's magnetic field was also carried out. The records for the months of October, December, March and May have been analysed, and the results of the analysis are given below. It is pointed out that there were relatively few observations in October, even after the 12th, and they were concentrated into the latter half of the month. The results, therefore, are not of the same reliability as those of December, March and May.

Observations at such a high latitude have seldom been carried out. The Oxford University Arctic Expedition 1935–1936 (Whatman and Hamilton 1938) made observations for 11 months in Northeastland, latitude 80° N., longitude 20° E., and the Louise A. Boyd Arctic Expedition obtained records on board the S.S. *Effie M. Morrissey* which sailed as far North as 78° in West Greenland waters during the summer of 1942.

It should be noted that the Oxford University Arctic Expedition observations were made at almost the same latitude and longitude as those at Barentsburg, but at quite different periods of the sunspot cycle.

All times in this paper are referred to G.M.T. Local noon was at 1100 G.M.T.

§ 2. EQUIPMENT AND SITE

(i) *Equipment*

The equipment used for taking the (h', f) observations was a very early model of Admiralty Equipment Type 249 (Thomas and Chalmers 1948), as developed at the National Physical Laboratory, and was, in fact, the first to undergo prolonged

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field trials. It possessed several features (some of which have been modified in later models) which must be borne in mind when reading the records and examining the photographic reproductions accompanying this paper. They are:

(*a*) The frequency range was 2·2–16·0 Mc/s., covered in two stages, 2·2–7·0 and 7·0–16·0 Mc/s. The lowest frequency was so high as often to be greater than f^0_E and sometimes greater than $f^0_{F_2}$.*

(*b*) The gain of the receivers increased from the low frequency end of each stage to the high. For this reason weak echoes might be quite clear at the high frequency end of the 2·2–7·0 Mc/s. stage, approaching 7·0 Mc/s., but quite invisible above 7·0 Mc/s.

(*c*) The bottom height marker line represented a value of apparent height, h', varying between 90 and 115 km., depending on the frequency being observed and the adjustment of the equipment. For the purposes of this report, the value has been taken as 100 km. The height marker lines are spaced 50 km. apart. The term height always means apparent height.

(*d*) Each complete (h', f) record was made in 10 seconds. This was a considerable advantage over any other equipment then existing, for it almost entirely eliminated errors caused by appreciable change in the ionization density during the taking of a record.

Power was obtained from a small petrol-electric generating set (2 kva.), which in general proved to be quite satisfactory, although the changes in voltage and frequency with varying load, inevitable with such a small power unit, necessitated very careful adjustment of the equipment and caused some irregularities on the records.

The magnetograph used for the observations of the earth's magnetic field proved unsatisfactory in many respects, of which the chief was that its sensitivity depended on the state of charge of the batteries used with it, and that there was no means of absolute calibration. Whilst there is no doubt, therefore, about the average diurnal trend of magnetic activity, the relative sizes of storms cannot be assessed with any certainty, and seasonal variation can only be determined by consideration of changes in the diurnal trend.

(ii) *Site*

All the equipment, except the aerials, was housed in a small hut of special construction, designed for polar conditions and made bullet and splinter proof. It was situated on a raised beach 200 ft. above sea level. The aerial site sloped gently down from East to West and, some 100 yards beyond, the ground fell steeply to the sea. To the East, the slope gradually increased to an escarpment 800 ft. high. The transmitting aerials of rhombic type were aligned North and South, and the receiving aerials of the same pattern East and West; the West mast was higher than the other outer masts to allow for the slope of the ground which during the period of observation was always frozen and was usually covered with snow.

Barentsburg lies well to the North of the zone of maximum auroral frequency. There is no ground sunrise for four months, 20th October to 20th February approximately, so that during this period each region of the upper atmosphere is cut off from the direct effect of ultra-violet light and uncharged corpuscular

* This notation is now superseded by fF_2, fxF_2, etc.

radiation, except for such radiation as has already passed through lower regions of the atmosphere.

The angle of dip of the earth's field is 81° and the gyro frequency 1·3 Mc/s. approximately.

The noise level was very low compared with lower latitudes and the effective sensitivity of the receivers was correspondingly increased.

§ 3. RESULTS

(i) *General*

The same main ionospheric regions are observed at Barentsburg as elsewhere. Ionospheric conditions, however, appear to change much more rapidly than at lower latitudes, and "abnormalities" are the rule rather than the exception. When the Northeastland observations, referred to above, were being made, it was realized that the interpretation of the records would be difficult, because of the changes in the ionosphere which occurred during the 20 minutes or so required to make each record. It was hoped that the records made with the Type 249 equipment (each taking virtually only five seconds because the echo pattern seldom extends beyond 7·0 Mc/s.) would be easier to read, and would clear up some of the points which were left in doubt by the Northeastland records. This has in fact proved to be the case. At the same time a number of new and unsuspected phenomena have been revealed, for which only a tentative explanation can be put forward until further data become available.

Despite the fact that the records are much easier to read than those made in Northeastland there are still many in which the value of the F region critical frequencies cannot be determined, quite apart from those records which show the "no echo condition" and those in which the F region is masked by Abnormal E.

(ii) *Magnetic Records*

The diurnal trend of magnetic activity is shown for December, March and May in Figure 1. The ordinates show the percentage of days in the month

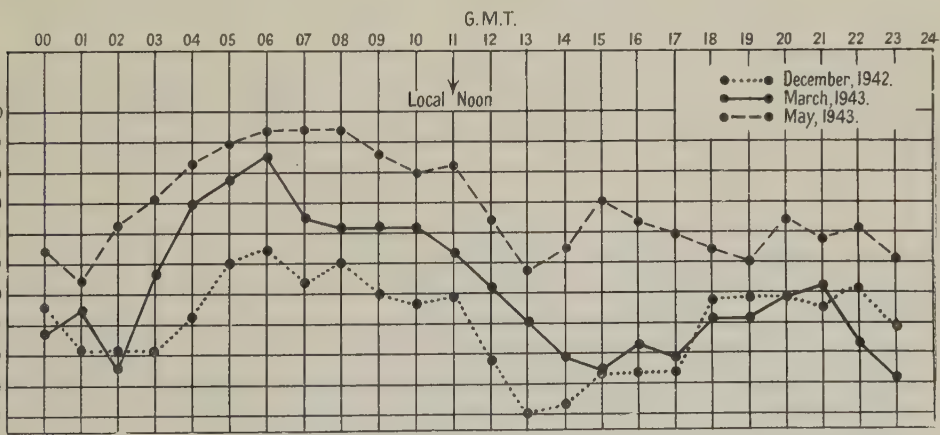

Figure 1. Percentage of time that a magnetic disturbance was present.

on which a disturbance was shown by the magnetograph for each hour. Only the shape of the curves is significant because of the variable sensitivity of the magnetograph and it is not possible to define what constitutes a disturbance, the

presence or absence of which has been a matter of judgment in reading the records. In each case there is a large maximum in the early morning, at about 06 hours, a less marked maximum in the afternoon, the actual time of maximum varying from month to month between 15 hours and 21 hours, and a well-defined minimum between 01 and 02 hours and between 13 and 15 hours. It may be mentioned that all the other months during which observations were made exhibit the same type of diurnal cycle with maxima at 06 hours and in the evening, and minima at 01 and 13 hours approximately.

When it has been necessary to examine correlation between magnetic activity and other phenomena these curves have been used and an additional test applied by selecting some 50 hours during each month when the records leave no doubt as to the presence of a magnetic storm and 100 hours during each month when conditions were certainly quiet, and comparing the individual (h', f) records with the corresponding magnetogram.

(iii) *The Normal E Region*

The criterion used for estimating f^0_E was that the lower frequency end of the F_1 echo pattern should show group retardation tending to infinity. It is considered that any other criterion, e.g. first appearance of an F echo, is unsatisfactory, owing to the possibility of absorption of the low frequency end of the F_1 echo pattern, and to the low gain of the receiver at low frequencies.

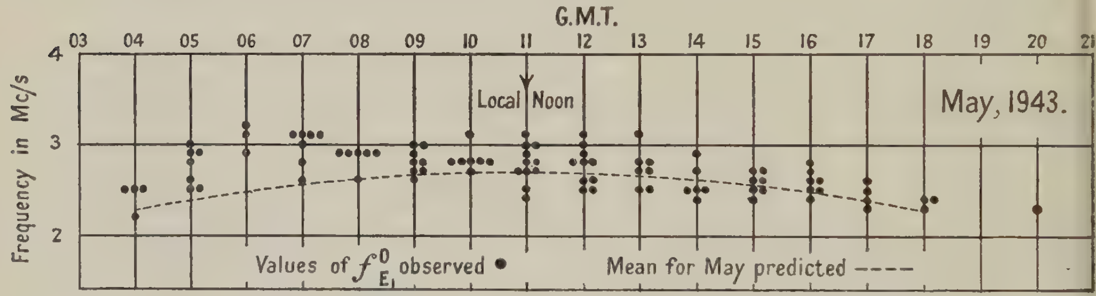

Figure 2.

The observed values of f^0_E for May are shown in Figure 2, together with the expected value based on the sun's zenith angle for mid-May. The agreement is good, but there is a tendency for the observed values to be high, particularly in the earlier part of the day.

(iv) *The Abnormal E Region*

Abnormal E ionization below a height of 140 km. appears to be of two types, one of which occurs very frequently. The common type of Abnormal E region has a low reflection coefficient, and reflections of a greater order than the second were never observed; it is often only a partial reflector, with F region echoes visible at the same time, and it does not usually show marked group retardation. The rare type of Abnormal E region has a high reflection coefficient, showing often up to seven reflections, and echoes usually persist to 10 Mc/s. or more; the F region is always completely or nearly completely obscured. This type of Abnormal E region occurred only between noon and midnight and most frequently at 18 hours, and never shows marked group retardation. The times and number of occurrences for each month are shown in the Table.

Occurrences of Abnormal E with High Reflection Coefficient

	14	15	16	17	18	19	20	21	22	23 hrs.	Totals
Oct. '42. (half-month)	1	1	2	1		1	1				7 (half-month)
Dec. '42.			1	1	3	2					7
Mar. '43.			1	4	2		1	1	1	1	11
May '43.				3	4	2	1	2	1	1	14
Totals	1	1	4	9	9	5	3	3	2	2	39

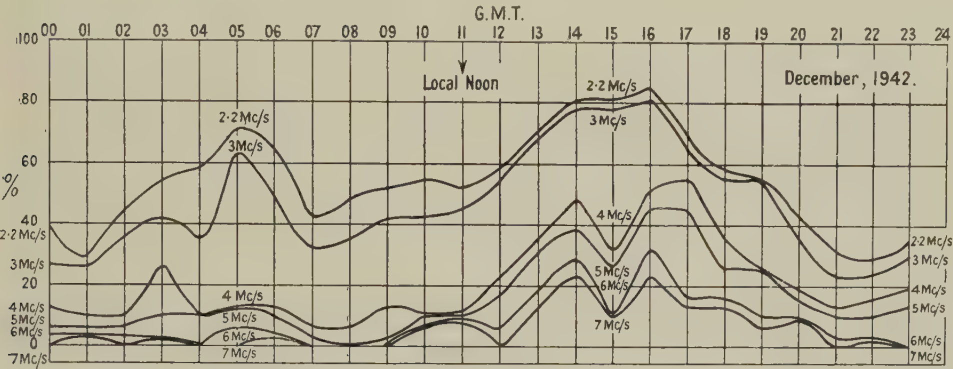

Figure 3. Percentage time maximum frequency of Abnormal E was not less than frequency indicated.

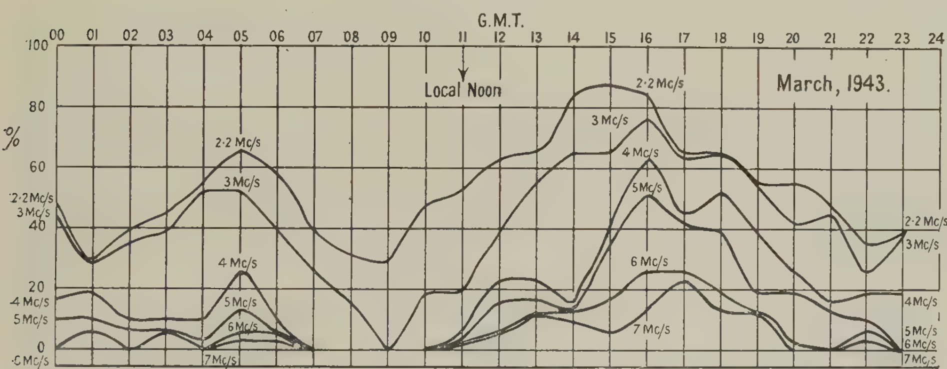

Figure 4. Percentage time maximum frequency of Abnormal E was not less than frequency indicated.

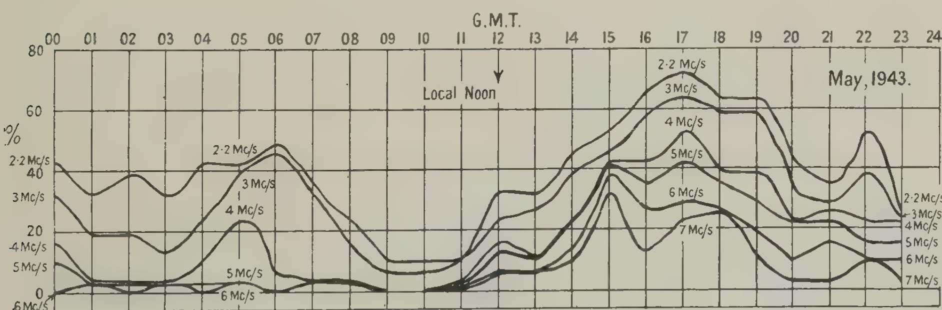

Figure 5. Percentage time maximum frequency of Abnormal E was not less than frequency indicated.

It appears from these results that it is of more frequent occurrence in summer than in winter and then occurs on approximately 14% of the days at 17–18 hours. The magnetic conditions can be either quiet or stormy. The Northeastland results show a maximum at 16 hours and that the "Intense E" then occurs on as many as 20% of the times observed, but only during quiet magnetic conditions. Referred to local times the maxima at Barentsburg and Northeastland are within an hour of each other.

In Figures 3, 4 and 5 are plotted for December, March and May for each hour the percentage of the observations for which the highest frequency of either type of Abnormal E echoes was not lower than the frequency indicated against each curve. The highest frequency at which Abnormal E echoes are observed is probably not independent of transmitter power.

In each case there is a maximum between 05 and 06 hours in the morning and a higher maximum between 14 and 17 hours in the evening; broad minima occur between 21 and 03 hours and between 07 and 10 hours, whatever frequency is considered. The results for October are precisely similar but there are not sufficient readings to justify a graph.

There does not appear to be any very marked seasonal variation; there is however a large occurrence of Abnormal E echoes to 3·0 Mc/s. in December, and the hours of maximum and minimum are one hour later in May than in December.

No correlation between Abnormal E of either type and magnetic activity has been found, and these results are quite different from those obtained in Northeastland, from the Louise A. Boyd observations and also from observations made in Alaska.

It is evident that Abnormal E reflection will govern oblique incidence transmission (i.e. determination of the maximum usable frequency, M.U.F.) for much of the time. But apart from the erratic variations of Abnormal E ionization with time at any given locality, it appears that there are probably large variations over fairly short distances at any given instant, so that predictions of M.U.F. are likely to be very unreliable.

(v) *The Normal E_2 Region*

A region intermediate between E and F_1 at a height of 150 km. approximately with the characteristics of a normal region, e.g. energy transference to F_1 at a definite point, with group retardation, is occasionally observed. There are not really sufficient readings to be certain that the values of $f^0_{E_2}$ are representative. In May, however, there are between four and eight readings for each hour from 11 hours to 15 hours inclusive, with average values 2·9, 3·0, 3·1, 3·1, 2·8 Mc/s. It will be noticed that the maximum occurs at 13–14 hours and not at local noon.

(vi) *The Abnormal E_2 Region*

There is often much abnormal ionization between the E and the F regions. This has been classed as Abnormal E_2. When there are two distinct Abnormal E regions present simultaneously, the higher is termed Abnormal E_2, although the height may be as low as 130 km. (that of Abnormal E then being say 80 km.). When a single Abnormal E region is present it is termed Abnormal E if its height is less than 150 km. and Abnormal E_2 if its height is between 150 and 190 km. Thus the height of Abnormal E may be between 80 and 140 km. and that of

Abnormal E_2 between 130 and 190 km. Abnormal E_2 usually shows no group retardation, but sometimes the group retardation is very marked and magneto-ionic splitting is shown, indicating a thick region. F region echoes are seldom obscured so that Abnormal E_2, even when thick, is only partially reflecting.

In Figures 6, 7 and 8 there is plotted for December, March and May for each hour the percentage of the observations for which the highest frequency of

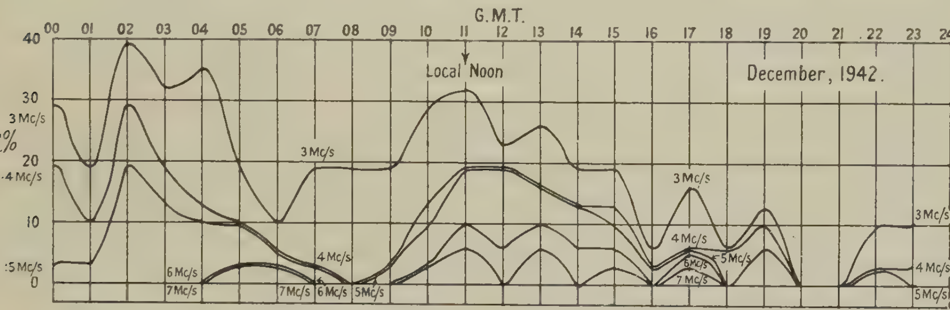

Figure 6. Percentage time maximum frequency of Abnormal E_2 was not less than frequency indicated.

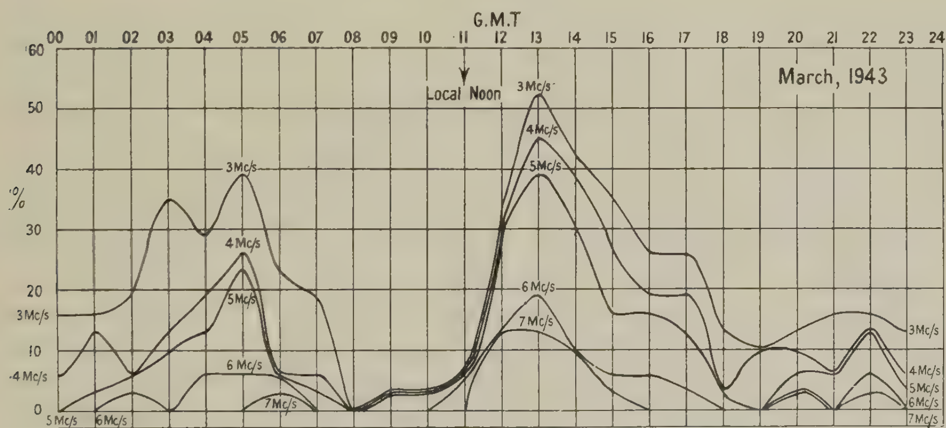

Figure 7. Percentage time maximum frequency of Abnormal E_2 was not less than frequency indicated.

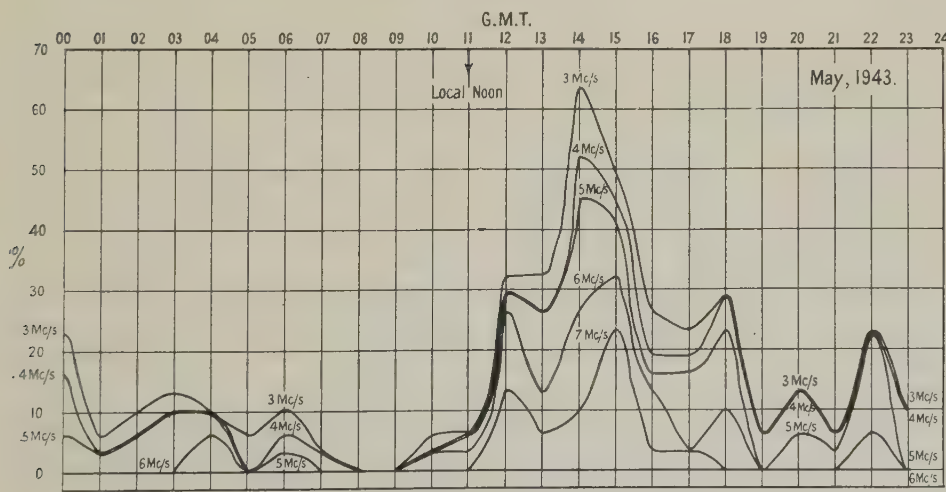

Figure 8. Percentage time maximum frequency Abnormal E_2 was not less than frequency indicated.

Abnormal E_2 echoes was not lower than the frequency indicated against each curve. The highest frequency at which Abnormal E_2 echoes are observed is probably not independent of transmitter power.

In December there are broad maxima at 02 and 12 hours and minima at 08 hours and 2030. In March the maxima are at 05 and 13 hours; one minimum is at 08 and a second very broad minimum between 19 and 24 hours. In May there is only one sharp maximum, at 14 hours, which is higher than in the other months. There is a minimum at 0830 with low readings at other times. The October results are very similar to those of March, but there are not sufficient readings to justify a graph.

As with the Abnormal E region, the time of afternoon maximum ionization is later—by two hours—in May than in December.

No correlation has been found with magnetic activity or Zenith aurora and these results are quite different from those obtained in Northeastland and from those obtained by the Louise A. Boyd expedition.

Polar echoes, as observed in Northeastland, did not occur.

(vii) *The Normal F_1 Region*

The normal F_1 region is absent throughout the winter and first appears at or near local noon towards the end of March; by mid-May it can be present throughout the 24 hours, but few complete days pass without a "no echo condition" or an echo pattern so chaotic that a regular F_1 region cannot be descried.

The average observed values of $f^{o}_{F_1}$, $f^{x}_{F_1}$, and the factor M 2,500 km. by which the critical frequency must be multiplied to obtain the M.U.F. for a transmission distance of 2,500 km., for May, are given in Figure 9. The factor was obtained by

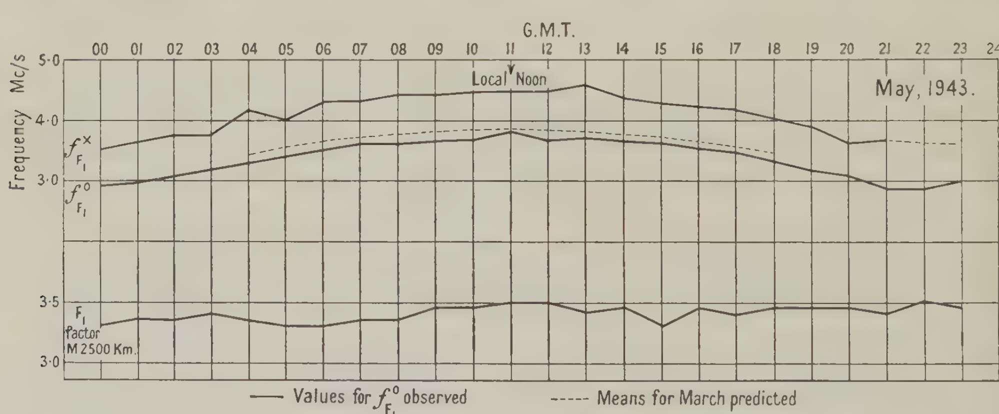

Figure 9. F_1 region.

the transmission curve slider technique (Millington 1938). The expected curve for $f^{o}_{F_1}$ based on the sun's zenith angle for mid-May is also given in the figure. The agreement between the observed and expected results is good, but all the observed values are a little lower than the expected.

The separation between the ordinary and extraordinary ray critical frequencies averages 0·74 Mc/s., the theoretical value being 0·71 Mc/s. The factor varies between 3·3 and 3·5 with its maximum value at local noon.

(viii) *Abnormalities in the F_1 Region*

There are two principal abnormalities in the F_1 region. The first, which is not very common—for example, there were only 27 cases in May—is that the F_1 region may consist of two distinct parts, each with a well defined critical frequency. An example of this is shown in Figure 10. The second, which is common, is that the F_1 region may exhibit marked spread which often persists to a frequency greater than $f^{o}_{F_2}$ and shows a marked critical frequency there. It is as if there were a normal winter F_2 record with h'_{F_2} at h'_{F_1}, superimposed on a normal summer F_1, F_2 record. An example is shown in Figure 11.

(ix) *The Normal F_2 Region*

The criteria for estimating the critical frequencies for the F_2 region were as follows: when the echo pattern is clear cut, as it often is in summer, there is no doubt about the value to take for the critical frequencies, because penetration takes place at a single frequency or over a very narrow band of frequencies. When the penetration took place over a wide band of frequencies, the higher frequency edge of this band was used, provided it was fairly sharply defined (see Figure 12). When the spread of penetration was such as to show no sharply defined high frequency edge, a point where most of the energy has penetrated was assumed to be the ordinary or extraordinary ray; sometimes this point is clearly shown by the second order reflection as in Figure 13. At other times it is necessary to make comparisons with records taken before and after in which perhaps there is no doubt as to the correct critical frequency.

It must be emphasized, however, that in winter there is quite a high percentage of records that cannot be analysed, and that the spread is usually very marked and often shows marked group retardation or tendencies toward critical frequencies. It is suggested that this spread may well be used for oblique angle transmission. Mr. R. Naismith has informed the writer that in England the spread will give signals of intensity some 30 db. below those from the true region, but in Spitsbergen, where spread is very strong, the drop in signal strength may be less. No experimental observations are available to check the possibility.

When there is a complete duplication of the F_2 echo pattern, as in Figure 14, the higher frequency pair was used for estimating critical frequencies.

In Figures 15, 16, 17 and 18 are plotted the values for $f^{o}_{F_2}$ and $f^{x}_{F_2}$ and also the highest frequency on which F echoes are observed, $f^{max}_{F_2}$. This latter gives a measure of the relative amount of spread and "polar spurs", which are described below, although the absolute value is no doubt dependent on transmitter power. The factor M 2,500 km. is also shown.

In October there is a marked diurnal variation in $f^{o}_{F_2}$ with a maximum at 09 hours and a minimum at 04 hours. The night time value of $f^{o}_{F_2}$ is of the same order as the December value when there is no sunrise. The frequency separation averages 0·74 Mc/s., the theoretical value being 0·72 Mc/s. There is considerable spread, and sensible echoes persist to much higher frequencies than $f^{x}_{F_2}$, particularly around local noon, when polar spurs are most numerous. The factor varies between 2·4 and 2·9.

In December $f^{o}_{F_2}$ shows little change throughout the 24 hours though there is a tendency for it to be higher after 12 hours than before. It is believed that, despite the difficulties of analysis, this curve gives a true measure of $f^{o}_{F_2}$, because the curve for $f^{x}_{F_2}$, estimated independently, follows the same shape. The frequency

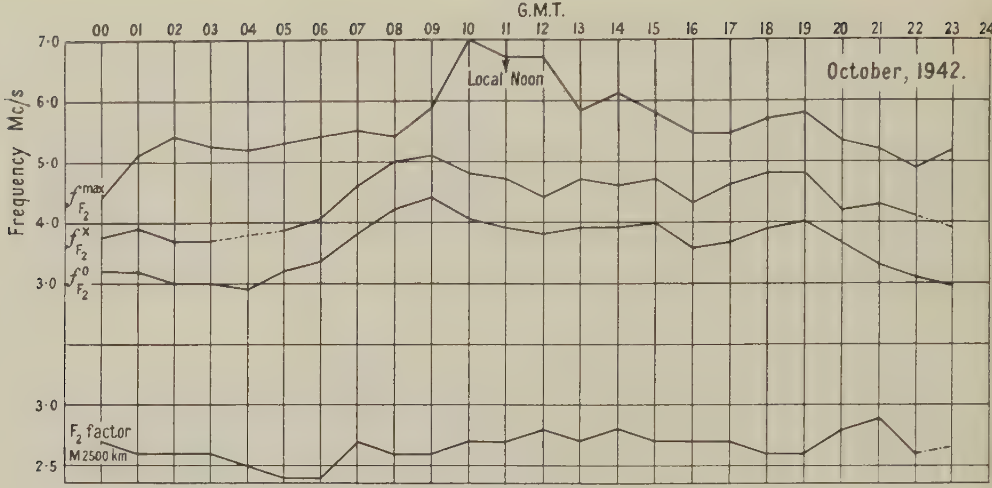

Figure 15. F_2 region.

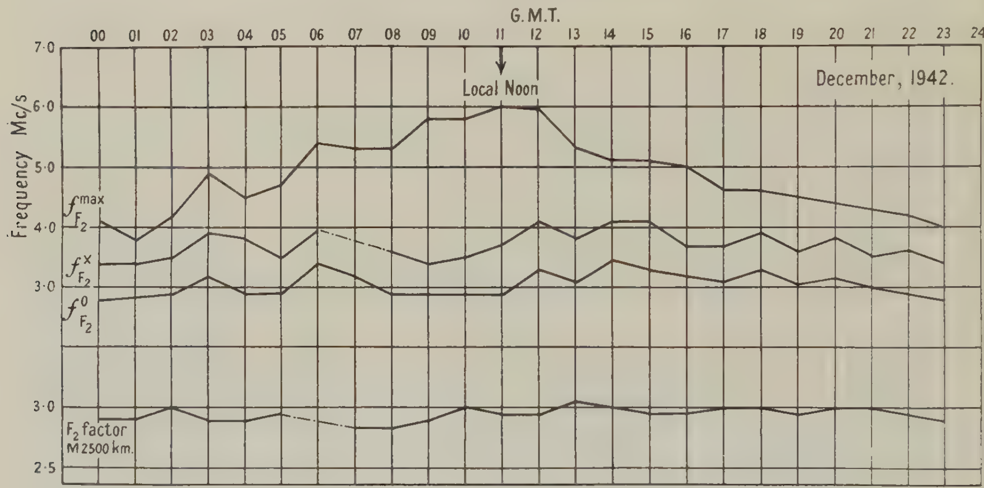

Figure 16. F_2 region.

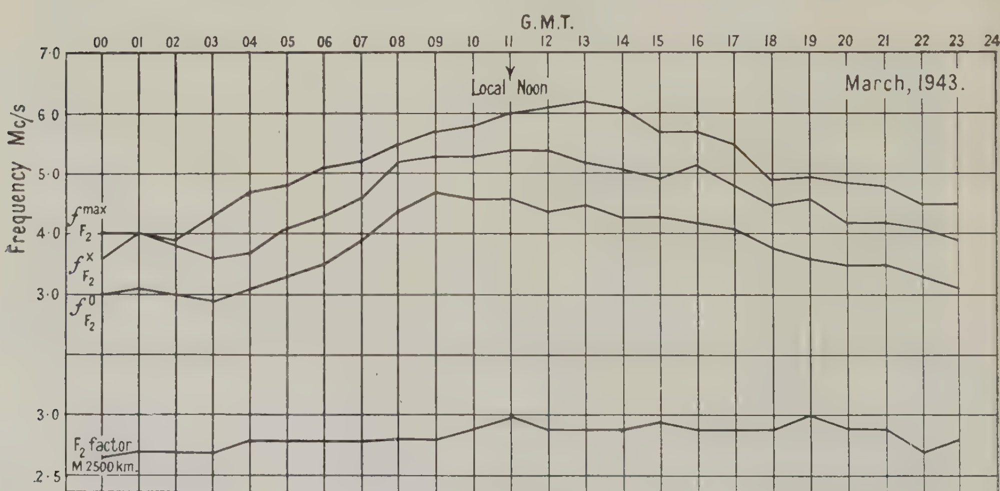

Figure 17. F_2 region.

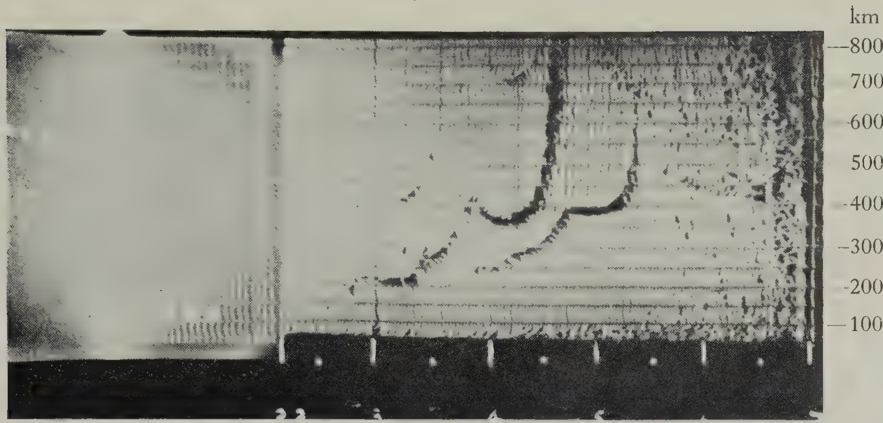

Figure 10. 1200 G.M.T. 22nd May 1943. The F_1 region consists of two distinct portions. Note the weak echo system starting at 3·2 Mc/s. at height 270 km.

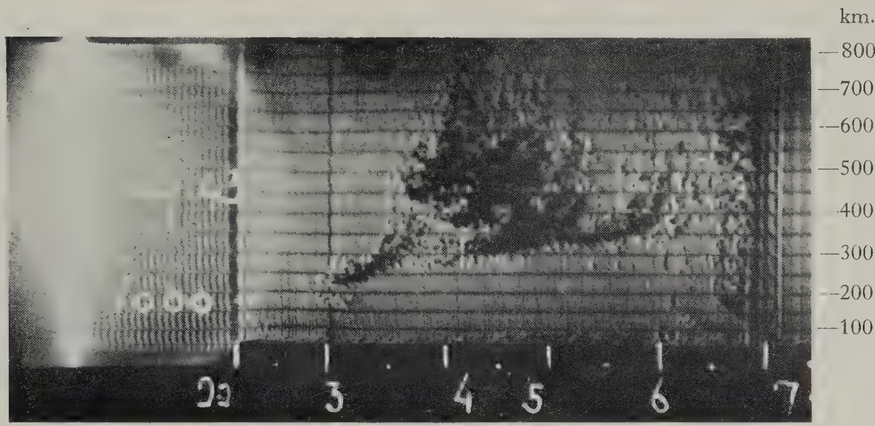

Figure 11. 1000 G.M.T. 27th May 1943. Strong spread at the F_1 level showing marked group retardation and critical point at 6·1 Mc/s.

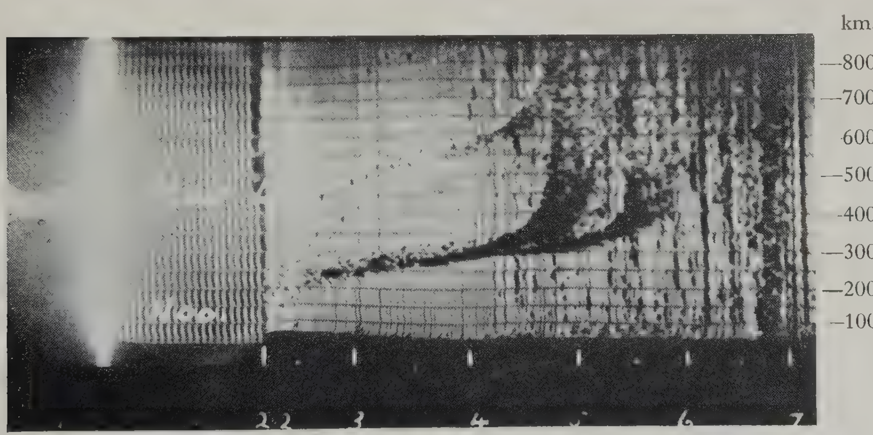

Figure 12. 1100 G.M.T. 26th March 1943. The penetration spreads over a band of frequencies, with clear-cut high frequency edge. $f^{\mathrm{O}}_{\mathrm{F}_2}$ is taken as 5·2 Mc/s. and $f^{\mathrm{X}}_{\mathrm{F}_2}$ as 5·9 Mc/s.

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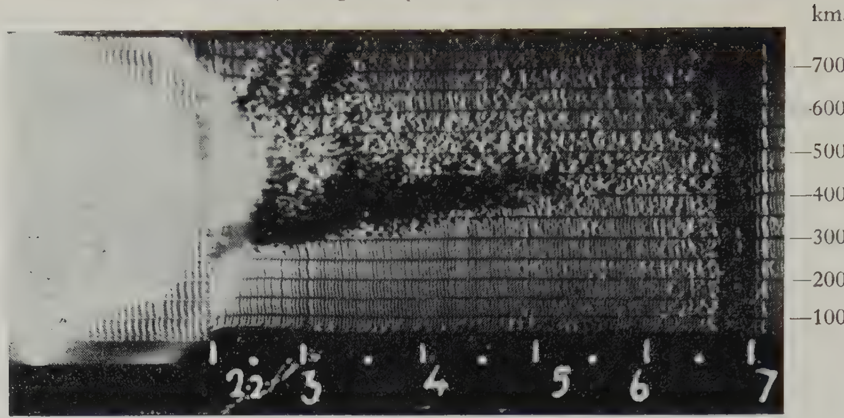

Figure 13. 1100 G.M.T. 4th February 1943. $f^{O}_{F_2}$ and $f^{X}_{F_2}$ are revealed by the second reflections.

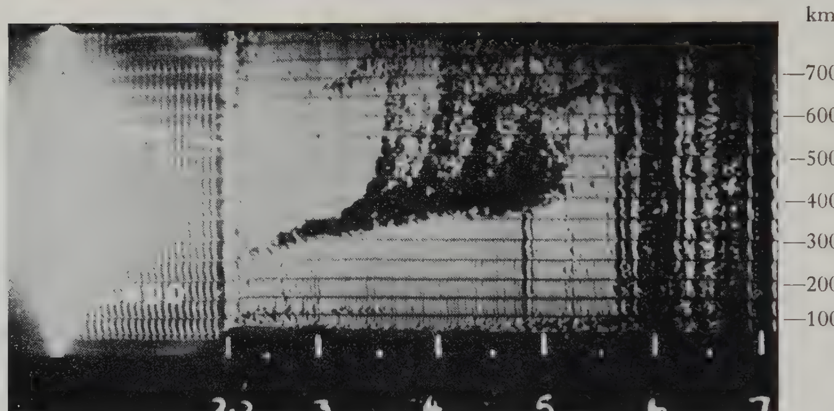

Figure 14. 2000 G.M.T. 16th May 1943. A duplicate set of F_2 echoes, the higher frequency pair penetrating to a higher region. $f^{O}_{F_2}$ is taken as 4·5 Mc/s. and $f^{X}_{F_2}$ as 5·1 Mc/s.

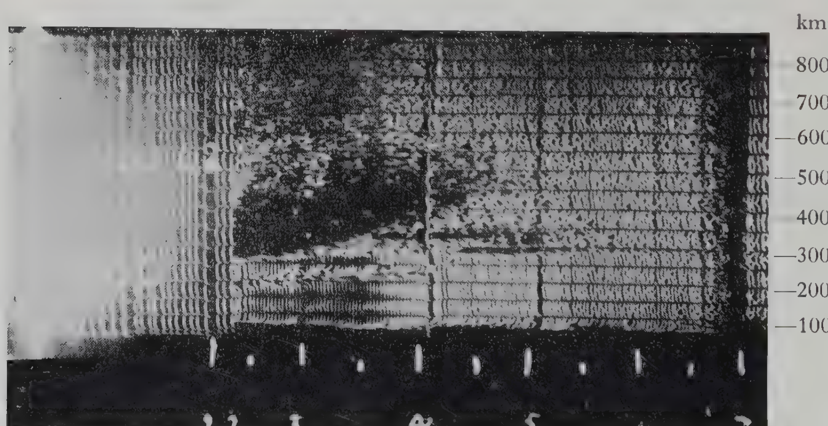

Figure 19. 1300 G.M.T. 17th November 1942. Polar spurs, one at height 280 km. extending to 7·0 Mc/s., another at height 310 km. extending to 5·5 Mc/s. There is also an abnormal E_2.

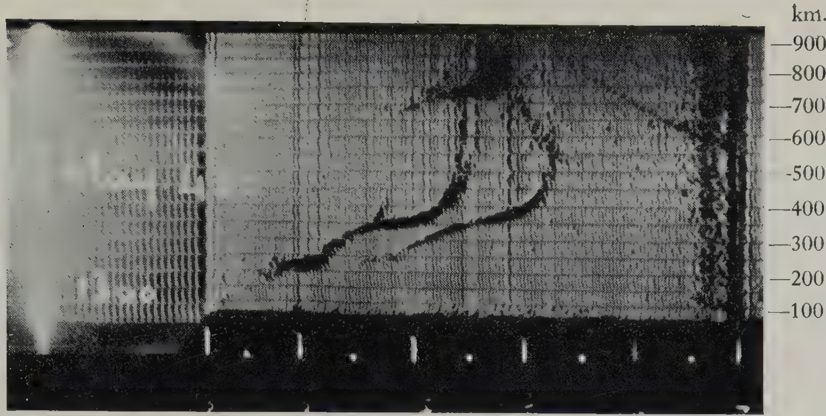

Figure 21. 1300 G.M.T. 7th May 1943. F_2 is triply split. There is also a higher region, not to be confused with the second reflection which is also visible.

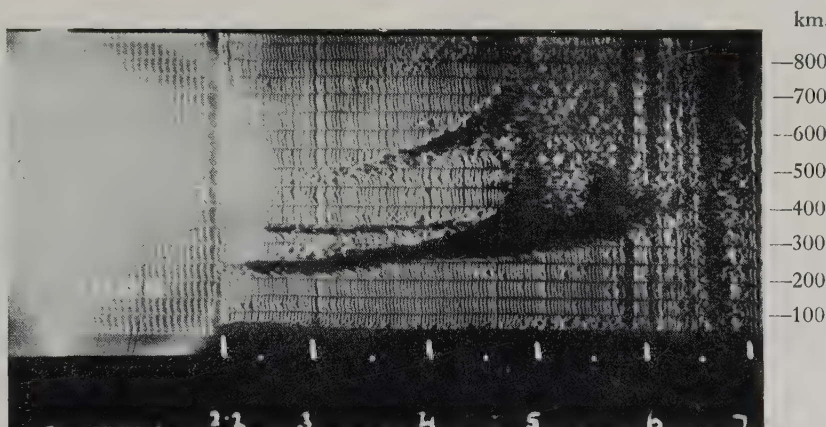

Figure 22. 1600 G.M.T. 24th March 1943. A higher region is seen from 2·6 Mc/s. at height 320 km.

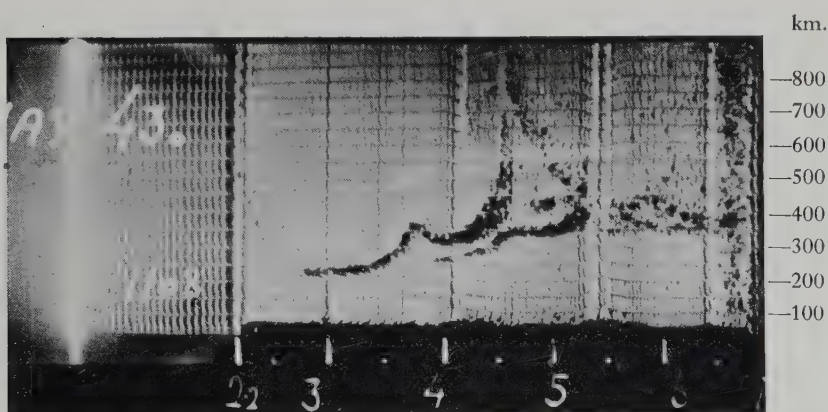

Figure 23. 1108 G.M.T. 7th May 1943. A higher region with very marked decrease in height as the frequency increases until a constant height is reached at 6·0 Mc/s.

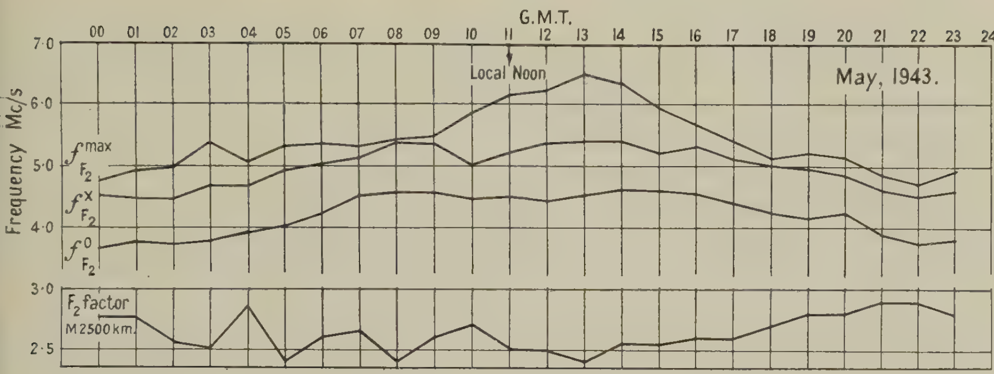

Figure 18. F_2 region.

separation averages 0·64 Mc/s., the theoretical value being 0·72 Mc/s. Echoes persist to much higher frequencies than $f^x_{F_2}$, particularly around local noon, at which time spurs are most numerous. The factor varies between 2·8 and 3·1, being higher in the afternoon than in the morning.

In March there is a marked diurnal variation in $f^o_{F_2}$ with a maximum at 09 hours and a minimum at 03 hours. The night time value of $f^o_{F_2}$ is of the same order as the December 24-hour value when there is no sunrise. The frequency separation averages 0·75 Mc/s., the theoretical value being 0·72 Mc/s. The spread is considerably reduced. The factor varies between 2·65 and 3·0, again being rather higher after 11 hours than before.

In May there is less diurnal variation in $f^o_{F_2}$, as might be expected, and the "night" time values are much higher than in March or December. There is no sunset at this time of the year. There are maxima at 08 and 14–15 hours. The frequency separation averages 0·77 Mc/s., the theoretical value being 0·70 Mc/s. Spread is very much reduced, and the spurs account for the wide separation between $f^x_{F_2}$ and $f^{max}_{F_2}$ between 10 and 15 hours. The factor varies between 2·4 and 2·9, being lowest when the F_1 region is most frequently present.

(x) *Abnormalities in the F_2 Region and above*

A phenomenon very frequently observed is the "polar spur". The presence of polar spurs was suspected in Northeastland and amply confirmed at Barentsburg with the Type 249 equipment, which is more suitable for their observation than that of the Oxford University Arctic Expedition. There is now no reason to suppose that they reflect only the extraordinary ray. A "polar spur" appears to be of a different nature from spurs reported by other workers and is best described pictorially by Figure 19. It is a thin echo system above, emerging from, or below the F region, and in nearly every case shows a constant or nearly constant height and no critical frequencies, the strength of the echoes gradually decreasing with increasing frequency. Very infrequently group retardation is exhibited by polar spurs or spurlike regions.

Polar spurs occur at all values of height from 200 km. up to at least 650 km. and it seems probable that they, Abnormal E_2 and Abnormal E are practically the same phenomenon occurring at different levels. They appear to be due to strata of high ionization density which are usually thin, as is shown by the lack of group retardation.

They are most numerous and extend to highest frequencies shortly after local noon. Figure 20 shows the occurrence of polar spurs and highest frequency to which they extend for May. There were 90 occurrences in December and 140 in March and May. The records available for October show relatively very few, so that there may be a seasonal effect with a maximum in spring and summer.

It is interesting that the times of afternoon maximum of frequency and occurrence of the polar spur, Abnormal E_2 and Abnormal E are in inverse order to their values of height, suggesting that the ionizing agency penetrates to lower levels of the atmosphere as time goes on. This, however, is not found to be universally true; for example, Abnormal E often appears before E_2 or a polar spur on individual days.

There are many occasions when echoes are visible from a distinct region above F_2 possessing "normal" characteristics—that is, it shows critical frequencies and perhaps ordinary and extraordinary rays. Heights for such regions vary between 300 and 700 km. It is not probable that these regions, which some

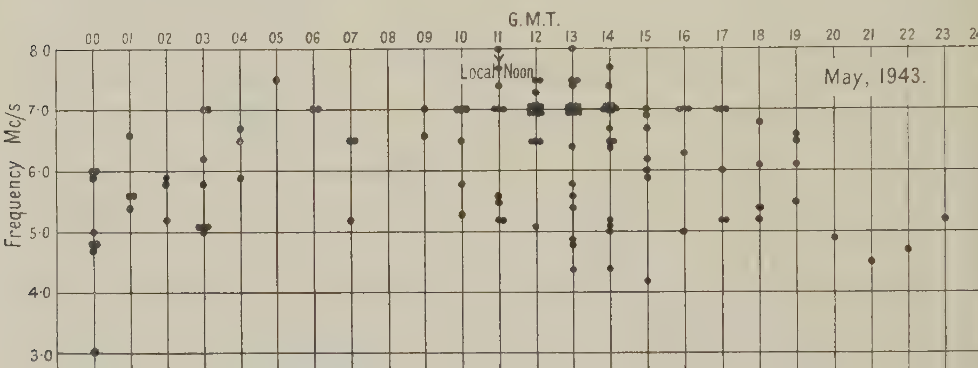

Figure 20. Maximum frequency of polar spurs.

workers have called "G", are always due to oblique angle radiation and reception, for the critical frequencies are seldom lower than those of F_2, and the energy often penetrates with group retardation from F_2 to the higher region. An example is shown in Figure 21. On the other hand, it not infrequently happens that echoes from the higher region are seen before $f^0_{F_2}$ (Figure 22). Apart from oblique transmission and reception, this can only be explained on the assumption that the F_2 region has holes in it sufficiently large to allow the passage of considerable quantities of energy to and from the higher region, or alternatively that it is partially reflecting. That one of these alternative explanations is correct is clearly shown by at least one record, where, after penetrating F_1, energy is returned simultaneously from F_2 and from a higher region, showing marked group retardation. A partially reflecting E region has long been known, but it does not appear likely that the much thicker F region can be partially reflecting, and the supposition of holes in it may be more probable.

On one occasion in March and on 17 occasions in May, mostly around local noon, an echo system was observed commencing at or near $f^0_{F_2}$ and descending in height with increasing frequency until it reached a height usually slightly greater than that of F_2, after which either it continued at a constant height like a polar spur or the echoes ceased. Like the usual type of polar spur, the echo system was thin and clear cut. If this is due to a thin stratum of ionization at a real height

nearly equal to apparent height, it is difficult to see why there should be such a large decrease in apparent height in passing beyond $f^{\circ}_{F_2}$ when the polar spurs show none. An example is shown in Figure 23.

It is theoretically possible for the F region to exhibit triple splitting, the third component being critical at a frequency lower than $f^{\circ}_{F_2}$ by an amount approximately equal to $f^{x}_{F_2} - f^{\circ}_{F_2}$. This third component is very weak, but it was observed infrequently in spring and summer (5 times in March and 6 times in May). An example is shown in Figure 21.

As was mentioned in subsection (ix) above, the F_2 region sometimes exhibits two complete sets of echoes with the same or nearly the same height (Figure 14). This is probably due to folds in the F region, so that reflection comes simultaneously from two points on the region.

(xi) *The No Echo Condition. Weak Echoes and Magnetic Storms*

The no echo condition was first reported by the British Polar Year party at Tromsø in 1932–33 (Appleton, Naismith and Ingram 1937). It was not observed in Northeastland at local noon or during the International day observations, although sudden cessation of echoes was sometimes observed on (h', t) records on 2·0 or 3·0 Mc/s. It was, however, frequently observed in Alaska, in Greenland and at Barentsburg. Signals from distant wireless stations can often be seen on the records during the no echo condition, showing that it is then of local occurrence. If it is assumed that no echoes would have been observed below 2·2 Mc/s., the lower recording limit, then the numbers of occasions the no echo condition was observed are: December 63, March 34, and May 33. There were many other occasions on which only weak Abnormal E or F echoes were observed.

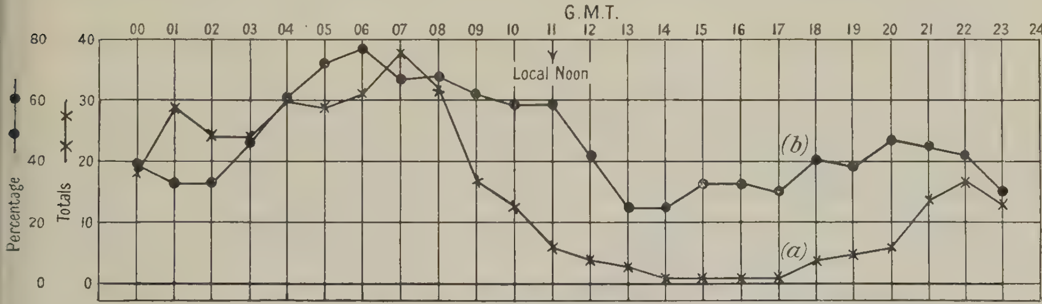

Figure 24. (*a*) Totals of no echo and weak echo conditions for December, March and May. (*b*) Percentage of time that a magnetic disturbance was present in December, March and May.

It is clear that the no echo condition and weak echoes are associated closely with magnetic storms, although the Dellinger effect may account for relatively few fades-out during daylight periods. The average diurnal magnetic activity curve is plotted together with the number of no echo and weak echo conditions in Figure 24, the results for December, March and May being summed together. Each month taken separately gives similar results. It is not permissible, however, to say that when there is a strong storm there will be no echoes or vice versa, although that is usually the case. Perhaps this may be accounted for by the fact that streams of charged particles which affect the earth's field may be situated over some part of the earth's surface far away from the observing station, and may

therefore produce no change in ionization directly overhead, and conversely charged particles overhead may produce no change in the vertical component of the field. During the dark period the no echo condition might be due to electron limitation. In December about 12% of the no echo and weak echo observations occurred without any apparent storm.

The writer made a very careful study of the Northeastland records obtained when zenith aurora was present, but no correlation between (h', f) curves and aurora was found. At Barentsburg zenith aurora does not appear to produce any definite effect either, although if any such effect was present it should be easy to see owing to the speed with which the records were taken.

§4. CONCLUSION

Time was not available during the war, and has not been found since, to analyse the records of the remaining months, although they have all been examined for points of special interest. It is thought, however, that the analysis of October, December, March and May, gives a good idea of ionospheric conditions in Spitsbergen during 1942–43.

In conclusion, the principal findings for each region may be summarized as follows:

Normal E obeys the sun's zenith angle law.

Abnormal E is very common with marked morning and evening maxima. No correlation with magnetic activity was found.

Normal E_2 is rare.

Abnormal E_2 is common with maxima in morning and evening except in May, when there was an evening maximum only. Abnormal E_2 with magneto-ionic splitting and group retardation is sometimes observed. No correlation with magnetic activity was found.

Normal F_1 obeys the sun's zenith angle law. Strong spread is often present at F_1 level.

Normal F_2. $f^{0}_{F_2}$ shows little diurnal change in December, but considerable diurnal change in October and March, with night time critical frequencies nearly the same as those in December. In May the diurnal change is again small with maximum values equal to those of March, but minimum values higher. Spread is very marked, especially in winter. The no echo condition and weak echoes are associated with magnetic activity.

ACKNOWLEDGMENTS

These observations were made on behalf of the Admiralty, with whose permission this paper is now published. The analysis was made in the autumn of 1943 at Baddow Research Laboratories, to the staff of which the writer is indebted for valuable criticism and advice.

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The Scotopic Visibility Function*

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Communicated by W. S. Stiles; MS. received 9th September 1948

ABSTRACT. The scotopic (dark-adaptation) visibility of radiation through the spectrum has been determined for fifty observers under 30 years of age. A modified photometric matching method was used at a very low brightness (3×10^{-6} candles/ft^2 or $3{\cdot}2 \times 10^{-9}$ stilbs). A subsidiary investigation showed that the ultimate scotopic curve was approximated closely. Another demonstrated the effect of age, showing a progressive decrease in sensitivity at the blue end of the spectrum with increasing age, the effect first becoming noticeable at about 30 years of age. A detailed comparison of the present results with those of previous workers is made and reasons for discrepancies are discussed.

§ 1. INTRODUCTION

THERE have been a number of determinations from time to time of the scotopic visibility function (luminosity function). The results are in general agreement, but there is sufficient difference between them to make it desirable to re-determine the function as accurately as possible so that reliable data are available for the calculation of effective filter transmissions, apparent brightness of coloured sources etc., at very low intensity levels.

§ 2. GENERAL EXPERIMENTAL CONDITIONS AND APPARATUS

(*a*) *Visual Conditions.* Both from the point of view of sensitivity, as well as of obtaining as nearly as possible the complete dark adaptation visibility curve, it was decided to use a large photometric field, the diameter subtending 20° at the eye, divided vertically into two equal parts. The intensity level was 3×10^{-6} candles/ft^2 ($3{\cdot}2 \times 10^{-9}$ stilbs), that is, about 15 times the absolute threshold. This was found to be the lowest level at which all observers could make true photometric settings without undue fatigue. There is no doubt that the ultimate scotopic visibility curve can only be obtained by a threshold method, but the conditions outlined above give a very close approximation to it. The approximation is rendered still closer by fixating on the upper edge of the field while making observations; the bulk of the field thus falls on the extra-foveal part of the retina. The magnitude of the small remaining difference between the visibility function thus obtained and the ultimate scotopic function determined by a threshold method has been measured for a small number of observers and found to be inappreciable except at the red end of the spectrum, where it is of the order of 10%.

In all measurements the full natural eye pupil was used, either completely unobstructed when using binocular vision, or looking through an exit pupil of at least 10 mm. diameter when using monocular vision.

* Communication from the National Physical Laboratory.

(*b*) *Apparatus.* A diagrammatic plan of the apparatus is shown in Figures 1 and 2. The second layout of apparatus was used in the main series of measurements in order that the visibility curve might be followed as far as possible, with the apparatus available, into the ends of the spectrum. All possible care was taken to eliminate stray light either by stray light filters (apparatus I) or by the double monochromator system (apparatus II).

For work in the blue end of the spectrum, the spectrometers in apparatus II were fitted with crown glass prisms for greater transparency in the violet and ultraviolet. The lower dispersion gave ample purity and, indeed, was an advantage, as a greater amount of light was transmitted for a given slit width.

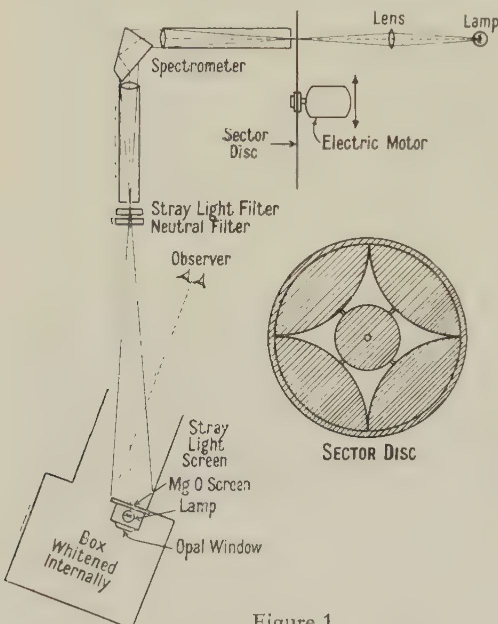

Figure 1.

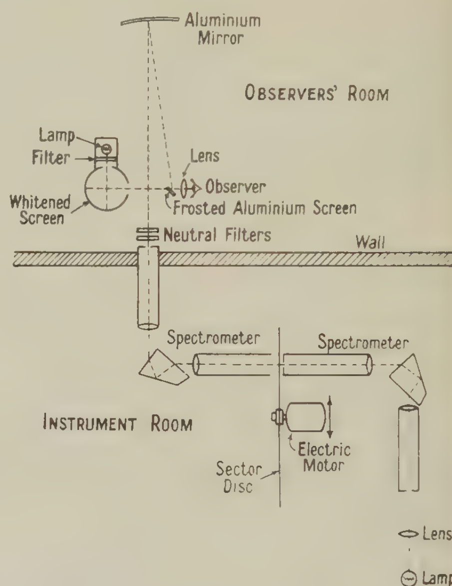

Figure 2.

The intensity of light emitted by the monochromator system was regulated by neutral filters and a sector disc with graded aperture which could be traversed across the beam. The form of sector aperture finally adopted is shown in the inset of Figure 1; this has the two advantages of being easy to make accurately, since all components are parts of circular discs which can be precisely machined, and of having an approximately logarithmic relation between displacement and transmission.

(*c*) *Physical Measurements.* The energy of radiation transmitted by the apparatus at each wavelength used was measured directly with a Schwarz thermopile in conjunction with a galvanometer amplifier. A Moll thermopile was also used as a check, except in the extreme blue. No systematic discrepancies were found between the two instruments. As the Moll thermopile was one of the standards of the Optics Section, Light Division, National Physical Laboratory, it is felt that this check demonstrates the reliability of the energy measurements.

During the period of experiments an energy calibration was made about once a month. It was not necessary to calibrate more frequently as the energy distribution of radiation emitted by the lamp and monochromator system changed very slowly.

The transmission factors of the neutral filters used were measured in position in the apparatus by means of a photocell (emission type) and amplifier. The filters all had a transmission of about 0·1, the requisite number being used to obtain the necessary total transmission. Transmissions of this order can be measured very accurately, and errors due to inter-reflections between the combined filters were avoided by tilting the filters very slightly with respect to each other.

(*d*) *Observers.* Preliminary tests with a variety of observers of both sexes and a wide range of ages indicated that with increasing age there was a decrease in sensitivity in the violet end of the spectrum (see Table 1). There appears

Table 1

Age Group:	under 20	21–30	31–40	41–50	over 51
Mean Age:	18·6	26·3	37·4	46·2	57·2
Wavelength (μ)	Log (Relative Visibility)				
0·41	$\bar{2}$·665	$\bar{2}$·714	$\bar{2}$·618	$\bar{2}$·554	$\bar{2}$·461
0·42	$\bar{1}$·066	$\bar{1}$·057	$\bar{2}$·991	$\bar{2}$·943	$\bar{2}$·850
0·44	$\bar{1}$·555	$\bar{1}$·525	$\bar{1}$·481	$\bar{1}$·445	$\bar{1}$·373
0·46	$\bar{1}$·826	$\bar{1}$·786	$\bar{1}$·762	$\bar{1}$·746	$\bar{1}$·700
0·48	$\bar{1}$·940	$\bar{1}$·921	$\bar{1}$·918	$\bar{1}$·914	$\bar{1}$·888
0·50	0·001	$\bar{1}$·992	$\bar{1}$·998	$\bar{1}$·992	$\bar{1}$·989
0·52	$\bar{1}$·953	$\bar{1}$·976	$\bar{1}$·979	$\bar{1}$·987	$\bar{1}$·985
0·54	$\bar{1}$·802	$\bar{1}$·820	$\bar{1}$·829	$\bar{1}$·854	$\bar{1}$·842
0·56	$\bar{1}$·508	$\bar{1}$·530	$\bar{1}$·533	$\bar{1}$·536	$\bar{1}$·540
0·58	$\bar{1}$·057	$\bar{1}$·056	$\bar{1}$·068	$\bar{1}$·074	$\bar{1}$·080
0·60	$\bar{2}$·536	$\bar{1}$·535	$\bar{2}$·539	$\bar{2}$·548	$\bar{2}$·547
0·62	$\bar{3}$·889	$\bar{3}$·903	$\bar{3}$·916	$\bar{3}$·925	$\bar{3}$·921
0·64	$\bar{3}$·191	$\bar{3}$·215	$\bar{3}$·213	$\bar{3}$·228	$\bar{3}$·247
0·66	$\bar{4}$·556	$\bar{4}$·567	$\bar{4}$·603	$\bar{4}$·590	$\bar{4}$·619
0·68	$\bar{5}$·995	$\bar{5}$·970	$\bar{5}$·980	$\bar{5}$·970	$\bar{5}$·972

to be little or no effect up to the age of 30, and so the main series of measurements was confined to observers under 30. Fifty observers were recruited, comprising twenty-five men and twenty-five women in case there should prove to be any systematic difference in scotopic sensitivity between the sexes. Most of them were inexperienced in photometric work, but it was found that accuracy and repetition depended more upon level of intelligence and enthusiastic co-operation than upon experience. Each observer was dark-adapted for one hour before taking readings, which occupied a further hour approximately according to the observer's speed of working. Observers were given a preliminary test on the Admiralty Adaptometer and rejected if below a third of the average sensitivity; only two or three observers failed to qualify in this way. Three observers (men) were also found to be partially colour blind, but this appeared to have no influence on the general shape of their scotopic visibility curves, and they were included with the rest.

§3. RESULTS

The results are given in full for all observers under 30 in Table 2 (binocular vision) and in Tables 3(*a*) and 3(*b*) (monocular vision). The proposed final mean figures are given in Table 4, derived from Tables 3(*a*) and 3(*b*).

Examination of the results in Tables 3(*a*) and 3(*b*) shows that the two sets do not link together perfectly. It was thought that this might be due to a seasonal

[*Continued on page* 330.

Table 2. First Series of Measurements. Binocular Observation

Wave-length (μ)	Men Observer: Age:	G.W. 27	J.W.C. 20	J.T. 28	M.A.L. 17	W.G.G. 29	K.H.S. 24	G.D.D. 26	G.G.T. 20	R.B.S. 26	Mean 24·1
		Log (Relative Visibility)									
0·41		$\bar{2}$·756	$\bar{2}$·763	$\bar{2}$·838	$\bar{2}$·689	$\bar{2}$·638	$\bar{2}$·594	$\bar{2}$·806	$\bar{2}$·672	$\bar{2}$·670	$\bar{2}$·714
0·42		$\bar{1}$·065	$\bar{1}$·072	$\bar{1}$·101	$\bar{1}$·142	$\bar{1}$·015	$\bar{1}$·015	$\bar{1}$·088	$\bar{1}$·078	$\bar{1}$·011	$\bar{1}$·059
0·44		$\bar{1}$·564	$\bar{1}$·537	$\bar{1}$·530	$\bar{1}$·609	$\bar{1}$·506	$\bar{1}$·495	$\bar{1}$·540	$\bar{1}$·541	$\bar{1}$·480	$\bar{1}$·534
0·46		$\bar{1}$·764	$\bar{1}$·809	$\bar{1}$·795	$\bar{1}$·832	$\bar{1}$·810	$\bar{1}$·773	$\bar{1}$·804	$\bar{1}$·796	$\bar{1}$·765	$\bar{1}$·795
0·48		$\bar{1}$·956	$\bar{1}$·924	$\bar{1}$·917	$\bar{1}$·941	$\bar{1}$·924	$\bar{1}$·918	$\bar{1}$·940	$\bar{1}$·944	$\bar{1}$·905	$\bar{1}$·930
0·50		$\bar{1}$·985	$\bar{1}$·992	$\bar{1}$·986	$\bar{1}$·985	0·003	$\bar{1}$·986	$\bar{1}$·996	0·003	0·009	$\bar{1}$·994
0·52		$\bar{1}$·975	$\bar{1}$·973	$\bar{1}$·982	$\bar{1}$·971	$\bar{1}$·976	$\bar{1}$·992	$\bar{1}$·983	$\bar{1}$·964	$\bar{1}$·950	$\bar{1}$·974
0·54		$\bar{1}$·767	$\bar{1}$·801	$\bar{1}$·874	$\bar{1}$·824	$\bar{1}$·822	$\bar{1}$·815	$\bar{1}$·832	$\bar{1}$·815	$\bar{1}$·811	$\bar{1}$·818
0·56		$\bar{1}$·493	$\bar{1}$·516	$\bar{1}$·523	$\bar{1}$·513	$\bar{1}$·528	$\bar{1}$·527	$\bar{1}$·525	$\bar{1}$·519	$\bar{1}$·505	$\bar{1}$·517
0·58		$\bar{1}$·029	$\bar{1}$·049	$\bar{1}$·094	$\bar{1}$·037	$\bar{1}$·093	$\bar{1}$·059	$\bar{1}$·035	$\bar{1}$·102	$\bar{1}$·000	$\bar{1}$·055
0·60		$\bar{2}$·521	$\bar{2}$·523	$\bar{2}$·600	$\bar{2}$·627	$\bar{2}$·535	$\bar{2}$·502	$\bar{2}$·504	$\bar{2}$·570	$\bar{2}$·504	$\bar{2}$·543
0·62		$\bar{3}$·928	$\bar{3}$·892	$\bar{3}$·923	$\bar{3}$·953	$\bar{3}$·910	$\bar{3}$·911	$\bar{3}$·921	$\bar{3}$·877	$\bar{3}$·887	$\bar{3}$·911
0·64		$\bar{3}$·217	$\bar{3}$·216	$\bar{3}$·259	$\bar{3}$·215	$\bar{3}$·194	$\bar{3}$·242	$\bar{3}$·243	$\bar{3}$·185	$\bar{3}$·215	$\bar{3}$·221
0·66		$\bar{4}$·536	$\bar{4}$·555	$\bar{4}$·616	$\bar{4}$·568	$\bar{4}$·534	$\bar{4}$·607	$\bar{4}$·614	$\bar{4}$·582	$\bar{4}$·614	$\bar{4}$·581
0·68		$\bar{5}$·882	$\bar{5}$·875	$\bar{4}$·096	$\bar{4}$·070	$\bar{5}$·871	$\bar{5}$·985	$\bar{4}$·015	$\bar{4}$·042	$\bar{5}$·974	$\bar{5}$·979

Table 2 (*cont.*)

Women — Log (Relative Visibility)

Wave-length (μ)	Observer: Age:	V.T. 26	S.M.I. 18	V.M.S. 30	V.T. 22	J.B.J. 18	M.J.G. 19	P.A.L. 18	M.W. 19	P.A.C. 20	Y.V.L. 18	J.B. 18	D.N.A. 22	D.P.C. 30	M.C.P. 18	N.H. 25	Mean 21·4
0·41		$\bar{2}$·775	$\bar{2}$·514	$\bar{2}$·606	$\bar{2}$·851	$\bar{2}$·801	$\bar{2}$·625	$\bar{2}$·546	$\bar{2}$·693	$\bar{2}$·714	$\bar{2}$·561	$\bar{2}$·586	$\bar{2}$·718	$\bar{2}$·657	$\bar{2}$·816	$\bar{2}$·621	$\bar{2}$·672
0·42		$\bar{1}$·093	$\bar{1}$·026	$\bar{1}$·073	$\bar{1}$·169	$\bar{1}$·147	$\bar{1}$·052	$\bar{1}$·024	$\bar{1}$·039	$\bar{1}$·044	$\bar{1}$·033	$\bar{1}$·014	$\bar{1}$·074	$\bar{1}$·032	$\bar{1}$·109	$\bar{2}$·995	$\bar{1}$·062
0·44		$\bar{1}$·507	$\bar{1}$·496	$\bar{1}$·441	$\bar{1}$·638	$\bar{1}$·559	$\bar{1}$·511	$\bar{1}$·613	$\bar{1}$·539	$\bar{1}$·564	$\bar{1}$·559	$\bar{1}$·559	$\bar{1}$·529	$\bar{1}$·496	$\bar{1}$·539	$\bar{1}$·555	$\bar{1}$·540
0·46		$\bar{1}$·791	$\bar{1}$·776	$\bar{1}$·703	$\bar{1}$·818	$\bar{1}$·826	$\bar{1}$·858	$\bar{1}$·839	$\bar{1}$·816	$\bar{1}$·864	$\bar{1}$·859	$\bar{1}$·762	$\bar{1}$·790	$\bar{1}$·784	$\bar{1}$·831	$\bar{1}$·802	$\bar{1}$·808
0·48		$\bar{1}$·907	$\bar{1}$·923	$\bar{1}$·940	$\bar{1}$·935	$\bar{1}$·954	$\bar{1}$·967	$\bar{1}$·927	$\bar{1}$·901	$\bar{1}$·953	$\bar{1}$·967	$\bar{1}$·940	$\bar{1}$·907	$\bar{1}$·905	$\bar{1}$·922	$\bar{1}$·894	$\bar{1}$·929
0·50		$\bar{1}$·973	0·001	0·015	$\bar{1}$·997	$\bar{1}$·998	0·010	$\bar{1}$·987	0·016	0·014	$\bar{1}$·981	0·021	$\bar{1}$·994	$\bar{1}$·972	0·002	$\bar{1}$·994	$\bar{1}$·998
0·52		$\bar{1}$·995	$\bar{1}$·981	$\bar{1}$·904	$\bar{1}$·987	$\bar{1}$·974	$\bar{1}$·933	$\bar{1}$·944	$\bar{1}$·968	$\bar{1}$·950	$\bar{1}$·923	$\bar{1}$·900	$\bar{1}$·974	$\bar{1}$·996	$\bar{1}$·957	$\bar{1}$·974	$\bar{1}$·957
0·54		$\bar{1}$·790	$\bar{1}$·815	$\bar{1}$·777	$\bar{1}$·819	$\bar{1}$·820	$\bar{1}$·780	$\bar{1}$·801	$\bar{1}$·800	$\bar{1}$·830	$\bar{1}$·809	$\bar{1}$·785	$\bar{1}$·849	$\bar{1}$·811	$\bar{1}$·761	$\bar{1}$·869	$\bar{1}$·808
0·56		$\bar{1}$·526	$\bar{1}$·492	$\bar{1}$·488	$\bar{1}$·599	$\bar{1}$·549	$\bar{1}$·459	$\bar{1}$·529	$\bar{1}$·529	$\bar{1}$·500	$\bar{1}$·505	$\bar{1}$·482	$\bar{1}$·522	$\bar{1}$·517	$\bar{1}$·481	$\bar{1}$·572	$\bar{1}$·517
0·58		$\bar{1}$·023	$\bar{1}$·094	$\bar{1}$·020	$\bar{1}$·143	$\bar{1}$·086	$\bar{1}$·013	$\bar{1}$·108	$\bar{1}$·082	$\bar{1}$·062	$\bar{1}$·073	$\bar{2}$·981	$\bar{1}$·034	$\bar{1}$·096	$\bar{1}$·016	$\bar{1}$·037	$\bar{1}$·058
0·60		$\bar{2}$·479	$\bar{2}$·497	$\bar{2}$·553	$\bar{2}$·609	$\bar{2}$·546	$\bar{2}$·457	$\bar{2}$·502	$\bar{2}$·516	$\bar{2}$·568	$\bar{2}$·536	$\bar{2}$·517	$\bar{2}$·492	$\bar{2}$·584	$\bar{2}$·562	$\bar{2}$·537	$\bar{2}$·530
0·62		$\bar{3}$·889	$\bar{3}$·893	$\bar{3}$·839	$\bar{3}$·944	$\bar{3}$·930	$\bar{3}$·890	$\bar{3}$·937	$\bar{3}$·878	$\bar{3}$·894	$\bar{3}$·814	$\bar{3}$·814	$\bar{3}$·874	$\bar{3}$·890	$\bar{3}$·888	$\bar{3}$·900	$\bar{3}$·885
0·64		$\bar{3}$·168	$\bar{3}$·258	$\bar{3}$·095	$\bar{3}$·297	$\bar{3}$·249	$\bar{3}$·123	$\bar{3}$·181	$\bar{3}$·169	$\bar{3}$·216	$\bar{3}$·139	$\bar{3}$·191	$\bar{3}$·183	$\bar{3}$·245	$\bar{3}$·215	$\bar{3}$·208	$\bar{3}$·196
0·66		$\bar{4}$·506	$\bar{4}$·657	$\bar{4}$·457	$\bar{4}$·632	$\bar{4}$·615	$\bar{4}$·548	$\bar{4}$·604	$\bar{4}$·496	$\bar{4}$·513	$\bar{4}$·489	$\bar{4}$·476	$\bar{4}$·532	$\bar{4}$·543	$\bar{4}$·568	$\bar{4}$·654	$\bar{4}$·553
0·68		$\bar{5}$·797	$\bar{5}$·978	$\bar{5}$·947	$\bar{4}$·107	$\bar{5}$·954	$\bar{4}$·123	$\bar{4}$·024	$\bar{5}$·942	$\bar{5}$·917	$\bar{5}$·926	$\bar{5}$·996	$\bar{5}$·996	$\bar{5}$·913	$\bar{4}$·094	$\bar{4}$·060	$\bar{5}$·985

Table 3(a). Second Series of Measurements. Monocular Observation

Men — Log (Relative Visibility)

Wave-length (μ)	Observer: Age:	K.H.S. 24	B.J. 25	R.G.B. 19	R.B.S. 26	W.G.G. 29	G.D.D. 26	G.G.T. 20	D.H.C. 22	W.E.B. 27	R.E.F.O. 22	E.H.M. 26	D.D.E. 22	R.F.B. 26
0·38		$\bar{4}$·520	$\bar{4}$·730	$\bar{4}$·145	$\bar{4}$·534	$\bar{4}$·136	$\bar{3}$·372	$\bar{4}$·785	$\bar{4}$·212	$\bar{4}$·838	$\bar{4}$·760	$\bar{5}$·972	$\bar{3}$·055	$\bar{4}$·473
0·40		$\bar{3}$·908	$\bar{2}$·000	$\bar{3}$·747	$\bar{3}$·973	$\bar{3}$·734	$\bar{2}$·334	$\bar{2}$·083	$\bar{3}$·766	$\bar{2}$·045	$\bar{2}$·050	$\bar{3}$·619	$\bar{2}$·116	$\bar{3}$·776
0·42		$\bar{2}$·995	$\bar{1}$·078	$\bar{2}$·997	$\bar{1}$·070	$\bar{2}$·960	$\bar{1}$·080	$\bar{1}$·028	$\bar{2}$·938	$\bar{2}$·952	$\bar{1}$·074	$\bar{2}$·915	$\bar{1}$·000	$\bar{2}$·902
0·44		$\bar{1}$·536	$\bar{1}$·510	$\bar{1}$·524	$\bar{1}$·536	$\bar{1}$·496	$\bar{1}$·540	$\bar{1}$·566	$\bar{1}$·484	$\bar{1}$·428	$\bar{1}$·513	$\bar{1}$·544	$\bar{1}$·494	$\bar{1}$·423
0·46		$\bar{1}$·737	$\bar{1}$·753	$\bar{1}$·775	$\bar{1}$·883	$\bar{1}$·725	$\bar{1}$·804	$\bar{1}$·759	$\bar{1}$·740	$\bar{1}$·718	$\bar{1}$·799	$\bar{1}$·825	$\bar{1}$·762	$\bar{1}$·809
0·48		$\bar{1}$·874	$\bar{1}$·890	$\bar{1}$·913	$\bar{1}$·969	$\bar{1}$·897	$\bar{1}$·934	$\bar{1}$·897	$\bar{1}$·820	$\bar{1}$·903	$\bar{1}$·912	$\bar{1}$·916	$\bar{1}$·935	$\bar{1}$·883
0·50		$\bar{1}$·987	$\bar{1}$·975	$\bar{1}$·971	0·006	$\bar{1}$·988	$\bar{1}$·996	$\bar{1}$·979	$\bar{1}$·947	$\bar{1}$·953	0·022	$\bar{1}$·998	$\bar{1}$·952	$\bar{1}$·969
0·52		$\bar{1}$·987	0·002	0·004	$\bar{1}$·968	$\bar{1}$·993	$\bar{1}$·974	0·002	0·001	0·007	$\bar{1}$·943	$\bar{1}$·976	0·005	0·008
0·54		$\bar{1}$·867	$\bar{1}$·876	$\bar{1}$·820	$\bar{1}$·871	$\bar{1}$·845	$\bar{1}$·866	$\bar{1}$·897	$\bar{1}$·908	$\bar{1}$·848	$\bar{1}$·861	$\bar{1}$·839	$\bar{1}$·880	$\bar{1}$·866
0·56		$\bar{1}$·586	$\bar{1}$·558	$\bar{1}$·616	$\bar{1}$·635	$\bar{1}$·599	$\bar{1}$·632	$\bar{1}$·624	$\bar{1}$·599	$\bar{1}$·579	$\bar{1}$·599	$\bar{1}$·624	$\bar{1}$·632	$\bar{1}$·626

Wave-length (μ)	Observer: Age:	D.W.B. 19	A.J.R. 19	M.A.L. 19	J.W.C. 21	E.H.R. 28	J.T. 28	G.C.D. 26	J.M.S. 23	E.N.B. 23	J.V.P. 23	W.T.B. 21	M.A.W. 22	Mean 23·4
0·38		$\bar{3}$·043	$\bar{4}$·758	$\bar{4}$·333	$\bar{4}$·904	$\bar{5}$·901	$\bar{3}$·234	$\bar{4}$·481	$\bar{3}$·012	$\bar{4}$·315	$\bar{4}$·049	$\bar{3}$·500	$\bar{4}$·000	$\bar{4}$·602
0·40		$\bar{2}$·091	$\bar{3}$·985	$\bar{3}$·705	$\bar{2}$·100	$\bar{3}$·572	$\bar{2}$·267	$\bar{3}$·909	$\bar{2}$·163	$\bar{3}$·854	$\bar{3}$·681	$\bar{2}$·327	$\bar{3}$·616	$\bar{3}$·937
0·42		$\bar{1}$·009	$\bar{1}$·015	$\bar{2}$·944	$\bar{1}$·020	$\bar{2}$·904	$\bar{1}$·030	$\bar{1}$·073	$\bar{1}$·047	$\bar{2}$·880	$\bar{2}$·753	$\bar{1}$·045	$\bar{2}$·886	$\bar{2}$·984
0·44		$\bar{1}$·473	$\bar{1}$·467	$\bar{1}$·499	$\bar{1}$·490	$\bar{1}$·435	$\bar{1}$·521	$\bar{1}$·510	$\bar{1}$·539	$\bar{1}$·452	$\bar{1}$·409	$\bar{1}$·503	$\bar{1}$·391	$\bar{1}$·491
0·46		$\bar{1}$·772	$\bar{1}$·730	$\bar{1}$·715	$\bar{1}$·756	$\bar{1}$·664	$\bar{1}$·736	$\bar{1}$·720	$\bar{1}$·726	$\bar{1}$·735	$\bar{1}$·654	$\bar{1}$·736	$\bar{1}$·689	$\bar{1}$·749
0·48		$\bar{1}$·889	$\bar{1}$·957	$\bar{1}$·837	$\bar{1}$·904	$\bar{1}$·885	$\bar{1}$·937	$\bar{1}$·954	$\bar{1}$·906	$\bar{1}$·926	$\bar{1}$·821	$\bar{1}$·898	$\bar{1}$·915	$\bar{1}$·903
0·50		$\bar{1}$·979	$\bar{1}$·982	0·007	0·001	$\bar{1}$·994	$\bar{1}$·969	$\bar{1}$·996	$\bar{1}$·993	0·001	$\bar{1}$·994	$\bar{1}$·952	$\bar{1}$·980	$\bar{1}$·984
0·52		$\bar{1}$·992	$\bar{1}$·993	$\bar{1}$·956	$\bar{1}$·967	$\bar{1}$·989	$\bar{1}$·994	$\bar{1}$·985	$\bar{1}$·995	$\bar{1}$·972	$\bar{1}$·984	0·009	$\bar{1}$·996	$\bar{1}$·988
0·54		$\bar{1}$·829	$\bar{1}$·870	$\bar{1}$·820	$\bar{1}$·862	$\bar{1}$·914	$\bar{1}$·931	$\bar{1}$·793	$\bar{1}$·903	$\bar{1}$·919	$\bar{1}$·797	$\bar{1}$·840	$\bar{1}$·884	$\bar{1}$·864
0·56		$\bar{1}$·603	$\bar{1}$·623	$\bar{1}$·569	$\bar{1}$·627	$\bar{1}$·629	$\bar{1}$·686	$\bar{1}$·696	$\bar{1}$·628	$\bar{1}$·716	$\bar{1}$·516	$\bar{1}$·562	$\bar{1}$·668	$\bar{1}$·617

Table 3(*a*) (*cont.*)

Women — Log (Relative Visibility)

Wave-length (μ)	Observer: Age:	V.L. 26	J.A.R. 26	S.M.I. 19	V.L. 19	V.T. 22	M.W. 19	P.A.C. 20	P.A.L. 19	N.S. 25	M.J.G. 19	P.F. 19	J.M. 22	S.P. 24
0·38		$\bar{4}$·996	$\bar{4}$·571	$\bar{4}$·254	$\bar{4}$·108	$\bar{3}$·077	$\bar{4}$·452	$\bar{4}$·675	$\bar{4}$·097	$\bar{4}$·194	$\bar{4}$·087	$\bar{4}$·426	$\bar{5}$·887	$\bar{3}$·170
0·40		$\bar{2}$·206	$\bar{3}$·957	$\bar{3}$·802	$\bar{3}$·521	$\bar{2}$·145	$\bar{3}$·962	$\bar{2}$·004	$\bar{3}$·723	$\bar{3}$·802	$\bar{3}$·749	$\bar{3}$·914	$\bar{3}$·566	$\bar{2}$·284
0·42		$\bar{1}$·061	$\bar{1}$·038	$\bar{2}$·964	$\bar{2}$·978	$\bar{1}$·029	$\bar{1}$·002	$\bar{2}$·942	$\bar{2}$·945	$\bar{2}$·935	$\bar{1}$·046	$\bar{1}$·083	$\bar{2}$·913	$\bar{1}$·105
0·44		$\bar{1}$·527	$\bar{1}$·544	$\bar{1}$·497	$\bar{1}$·455	$\bar{1}$·470	$\bar{1}$·517	$\bar{1}$·503	$\bar{1}$·542	$\bar{1}$·501	$\bar{1}$·577	$\bar{1}$·563	$\bar{1}$·452	$\bar{1}$·488
0·46		$\bar{1}$·736	$\bar{1}$·775	$\bar{1}$·731	$\bar{1}$·760	$\bar{1}$·749	$\bar{1}$·813	$\bar{1}$·758	$\bar{1}$·804	$\bar{1}$·735	$\bar{1}$·812	$\bar{1}$·808	$\bar{1}$·849	$\bar{1}$·769
0·48		$\bar{1}$·864	$\bar{1}$·850	$\bar{1}$·904	$\bar{1}$·927	$\bar{1}$·919	$\bar{1}$·928	$\bar{1}$·892	$\bar{1}$·885	$\bar{1}$·900	$\bar{1}$·975	$\bar{1}$·948	$\bar{1}$·925	$\bar{1}$·917
0·50		$\bar{1}$·953	$\bar{1}$·961	$\bar{1}$·987	$\bar{1}$·946	$\bar{1}$·999	$\bar{1}$·975	0·010	0·008	0·012	$\bar{1}$·999	$\bar{1}$·985	0·012	$\bar{1}$·968
0·52		0·005	0·006	$\bar{1}$·998	0·002	$\bar{1}$·938	0·006	$\bar{1}$·950	$\bar{1}$·961	$\bar{1}$·932	$\bar{1}$·970	$\bar{1}$·992	$\bar{1}$·932	0·013
0·54		$\bar{1}$·861	$\bar{1}$·966	$\bar{1}$·902	$\bar{1}$·801	$\bar{1}$·816	$\bar{1}$·909	$\bar{1}$·891	$\bar{1}$·836	$\bar{1}$·842	$\bar{1}$·887	$\bar{1}$·833	$\bar{1}$·792	$\bar{1}$·866
0·56		$\bar{1}$·566	$\bar{1}$·504	$\bar{1}$·574	$\bar{1}$·543	$\bar{1}$·561	$\bar{1}$·634	$\bar{1}$·611	$\bar{1}$·606	$\bar{1}$·618	$\bar{1}$·579	$\bar{1}$·595	$\bar{1}$·601	$\bar{1}$·631

Wave-length (μ)	Observer: Age:	E.B.U. 23	M.R. 17	A.H. 27	D.J. 20	J.B.J. 25	R.H. 21	J.E.H. 22	E.N.P. 24	E.M.C. 19	E.P.C. 20	M.R.S. 23	K.M.B. 21	Mean 21·6
0·38		$\bar{4}$·974	$\bar{4}$·605	$\bar{3}$·077	$\bar{3}$·036	$\bar{4}$·360	$\bar{4}$·742	$\bar{5}$·493	$\bar{4}$·639	$\bar{4}$·962	$\bar{4}$·723	$\bar{3}$·040	$\bar{4}$·458	$\bar{4}$·564
0·40		$\bar{2}$·058	$\bar{3}$·942	$\bar{2}$·124	$\bar{2}$·071	$\bar{3}$·733	$\bar{3}$·922	$\bar{3}$·470	$\bar{2}$·118	$\bar{2}$·157	$\bar{3}$·950	$\bar{2}$·162	$\bar{3}$·971	$\bar{3}$·933
0·42		$\bar{2}$·980	$\bar{2}$·933	$\bar{2}$·978	$\bar{1}$·050	$\bar{2}$·937	$\bar{2}$·874	$\bar{2}$·914	$\bar{1}$·025	$\bar{1}$·021	$\bar{1}$·006	$\bar{1}$·047	$\bar{1}$·005	$\bar{2}$·992
0·44		$\bar{1}$·480	$\bar{1}$·459	$\bar{1}$·447	$\bar{1}$·489	$\bar{1}$·508	$\bar{1}$·400	$\bar{1}$·489	$\bar{1}$·621	$\bar{1}$·422	$\bar{1}$·498	$\bar{1}$·507	$\bar{1}$·492	$\bar{1}$·498
0·46		$\bar{1}$·723	$\bar{1}$·702	$\bar{1}$·728	$\bar{1}$·734	$\bar{1}$·741	$\bar{1}$·766	$\bar{1}$·781	$\bar{1}$·800	$\bar{1}$·743	$\bar{1}$·762	$\bar{1}$·749	$\bar{1}$·714	$\bar{1}$·762
0·48		$\bar{1}$·875	$\bar{1}$·886	$\bar{1}$·853	$\bar{1}$·960	$\bar{1}$·919	$\bar{1}$·787	$\bar{1}$·890	$\bar{1}$·922	$\bar{1}$·888	$\bar{1}$·895	$\bar{1}$·887	$\bar{1}$·843	$\bar{1}$·898
0·50		0·006	0·009	$\bar{1}$·927	$\bar{1}$·998	$\bar{1}$·999	$\bar{1}$·970	0·014	$\bar{1}$·982	$\bar{1}$·977	$\bar{1}$·987	$\bar{1}$·982	$\bar{1}$·970	$\bar{1}$·985
0·52		$\bar{1}$·928	$\bar{1}$·980	0·018	$\bar{1}$·969	$\bar{1}$·987	0·020	$\bar{1}$·968	$\bar{1}$·987	$\bar{1}$·999	$\bar{1}$·986	$\bar{1}$·985	$\bar{1}$·996	$\bar{1}$·981
0·54		$\bar{1}$·843	$\bar{1}$·900	$\bar{1}$·896	$\bar{1}$·839	$\bar{1}$·886	$\bar{1}$·824	$\bar{1}$·836	$\bar{1}$·882	$\bar{1}$·896	$\bar{1}$·879	$\bar{1}$·854	$\bar{1}$·906	$\bar{1}$·866
0·56		$\bar{1}$·575	$\bar{1}$·632	$\bar{1}$·639	$\bar{1}$·600	$\bar{1}$·621	$\bar{1}$·535	$\bar{1}$·622	$\bar{1}$·660	$\bar{1}$·630	$\bar{1}$·631	$\bar{1}$·632	$\bar{1}$·674	$\bar{1}$·603

Table 3(b). Third Series of Measurements. Monocular Observation

Men

Log (Relative Visibility)

Wave-length (μ)	Observer: Age:	W.T.B. 21	M.A.W. 22	J.V.P. 23	J.M.S. 23	J.W.C. 21	G.G.T. 20	W.G.G. 29	G.D.D. 26	W.E.B. 27	A.J.R. 19	G.C.D. 26	E.J.G. 27	K.C.N. 23
0·48		$\bar{1}$·855	$\bar{1}$·858	$\bar{1}$·717	$\bar{1}$·867	$\bar{1}$·791	$\bar{1}$·891	$\bar{1}$·828	$\bar{1}$·841	$\bar{1}$·745	$\bar{1}$·805	$\bar{1}$·755	$\bar{1}$·867	$\bar{1}$·707
0·50		0·005	$\bar{1}$·995	0·032	$\bar{1}$·990	$\bar{1}$·998	$\bar{1}$·968	0·016	$\bar{1}$·972	$\bar{1}$·964	$\bar{1}$·873	0·008	$\bar{1}$·996	$\bar{1}$·888
0·52		$\bar{1}$·967	$\bar{1}$·955	$\bar{1}$·890	$\bar{1}$·976	$\bar{1}$·943	$\bar{1}$·985	$\bar{1}$·886	$\bar{1}$·989	$\bar{1}$·995	0·040	$\bar{1}$·926	$\bar{1}$·969	0·038
0·54		$\bar{1}$·698	$\bar{1}$·807	$\bar{1}$·722	$\bar{1}$·806	$\bar{1}$·772	$\bar{1}$·742	$\bar{1}$·712	$\bar{1}$·812	$\bar{1}$·791	$\bar{1}$·835	$\bar{1}$·888	$\bar{1}$·777	$\bar{1}$·946
0·56		$\bar{1}$·466	$\bar{1}$·542	$\bar{1}$·413	$\bar{1}$·487	$\bar{1}$·512	$\bar{1}$·484	$\bar{1}$·452	$\bar{1}$·521	$\bar{1}$·671	$\bar{1}$·413	$\bar{1}$·547	$\bar{1}$·510	$\bar{1}$·355
0·58		$\bar{1}$·052	$\bar{1}$·141	$\bar{1}$·095	$\bar{1}$·044	$\bar{1}$·029	$\bar{1}$·052	$\bar{1}$·050	$\bar{1}$·132	$\bar{1}$·136	$\bar{1}$·082	$\bar{1}$·147	$\bar{1}$·086	$\bar{1}$·111
0·60		$\bar{2}$·568	$\bar{2}$·594	$\bar{2}$·495	$\bar{2}$·535	$\bar{2}$·521	$\bar{2}$·572	$\bar{2}$·484	$\bar{2}$·600	$\bar{2}$·757	$\bar{2}$·535	$\bar{2}$·561	$\bar{2}$·555	$\bar{2}$·602
0·62		$\bar{3}$·908	$\bar{3}$·948	$\bar{3}$·848	$\bar{3}$·893	$\bar{3}$·815	$\bar{3}$·874	$\bar{3}$·877	$\bar{3}$·908	$\bar{3}$·916	$\bar{3}$·864	$\bar{2}$·045	$\bar{3}$·966	$\bar{3}$·923
0·64		$\bar{3}$·171	$\bar{3}$·247	$\bar{3}$·164	$\bar{3}$·264	$\bar{3}$·116	$\bar{3}$·217	$\bar{3}$·086	$\bar{3}$·265	$\bar{3}$·367	$\bar{3}$·079	$\bar{3}$·292	$\bar{3}$·222	$\bar{3}$·537
0·66		$\bar{4}$·560	$\bar{4}$·614	$\bar{4}$·634	$\bar{4}$·650	$\bar{4}$·490	$\bar{4}$·695	$\bar{4}$·567	$\bar{4}$·715	$\bar{4}$·781	$\bar{4}$·601	$\bar{4}$·759	$\bar{4}$·646	$\bar{3}$·039
0·68		$\bar{5}$·874	$\bar{4}$·003	$\bar{4}$·021	$\bar{4}$·017	$\bar{5}$·853	$\bar{4}$·059	$\bar{5}$·813	$\bar{4}$·052	$\bar{5}$·931	$\bar{5}$·982	$\bar{5}$·948	$\bar{5}$·976	$\bar{4}$·378
0·70		$\bar{5}$·294	$\bar{5}$·312	$\bar{5}$·291	$\bar{5}$·324	$\bar{5}$·153	$\bar{5}$·329	$\bar{5}$·206	$\bar{5}$·338	$\bar{5}$·314	$\bar{5}$·297	$\bar{5}$·309	$\bar{5}$·287	$\bar{5}$·713
0·72		$\bar{6}$·754	$\bar{6}$·700	$\bar{6}$·925	$\bar{6}$·770	$\bar{6}$·608	$\bar{6}$·787	$\bar{6}$·623	$\bar{6}$·770	$\bar{6}$·793	$\bar{6}$·619	$\bar{6}$·822	$\bar{6}$·751	$\bar{5}$·185
0·74		$\bar{6}$·146	$\bar{6}$·124	$\bar{6}$·191	$\bar{6}$·203	$\bar{6}$·077	$\bar{6}$·242	$\bar{6}$·085	$\bar{6}$·225	$\bar{6}$·260	$\bar{6}$·175	$\bar{6}$·317	$\bar{6}$·148	$\bar{6}$·744
0·76		$\bar{7}$·625	$\bar{7}$·608	$\bar{7}$·559	$\bar{7}$·651	$\bar{7}$·489	$\bar{7}$·716	$\bar{7}$·536	$\bar{7}$·717	$\bar{7}$·683	$\bar{7}$·720	$\bar{7}$·706	$\bar{7}$·611	$\bar{6}$·032
0·78		$\bar{7}$·122	$\bar{7}$·131	$\bar{7}$·071	$\bar{7}$·138	$\bar{7}$·021	$\bar{7}$·162	$\bar{8}$·901	$\bar{7}$·110	$\bar{7}$·282	$\bar{7}$·046	$\bar{7}$·279	$\bar{7}$·116	$\bar{7}$·700

Wave-length (μ)	Observer: Age:	J.H.K. 19	E.H.M. 26	D.D.E. 22	R.E.F.O. 22	J.B. 20	M.A.L. 18	J.T. 28	E.N.B. 23	D.W.B. 19	R.F.B. 26	R.G.B. 19	B.J. 25	Mean 23·0
0·48		$\bar{1}$·815	$\bar{1}$·895	$\bar{1}$·900	$\bar{1}$·887	$\bar{1}$·849	$\bar{1}$·839	$\bar{1}$·826	$\bar{1}$·843	$\bar{1}$·816	$\bar{1}$·922	$\bar{1}$·844	$\bar{1}$·855	$\bar{1}$·853
0·50		$\bar{1}$·998	$\bar{1}$·925	$\bar{1}$·936	$\bar{1}$·996	$\bar{1}$·991	$\bar{1}$·989	$\bar{1}$·991	$\bar{1}$·906	$\bar{1}$·982	$\bar{1}$·987	$\bar{1}$·977	$\bar{1}$·994	$\bar{1}$·974
0·52		$\bar{1}$·962	0·035	0·018	$\bar{1}$·979	$\bar{1}$·946	$\bar{1}$·975	$\bar{1}$·993	0·032	$\bar{1}$·961	$\bar{1}$·977	$\bar{1}$·986	$\bar{1}$·975	$\bar{1}$·976
0·54		$\bar{1}$·768	$\bar{1}$·760	$\bar{1}$·741	$\bar{1}$·855	$\bar{1}$·748	$\bar{1}$·738	$\bar{1}$·811	$\bar{1}$·823	$\bar{1}$·795	$\bar{1}$·709	$\bar{1}$·710	$\bar{1}$·731	$\bar{1}$·780
0·56		$\bar{1}$·461	$\bar{1}$·388	$\bar{1}$·545	$\bar{1}$·527	$\bar{1}$·456	$\bar{1}$·501	$\bar{1}$·557	$\bar{1}$·435	$\bar{1}$·541	$\bar{1}$·611	$\bar{1}$·442	$\bar{1}$·490	$\bar{1}$·493
0·58		$\bar{1}$·263	$\bar{1}$·053	$\bar{1}$·055	$\bar{1}$·155	$\bar{1}$·027	$\bar{1}$·064	$\bar{1}$·077	$\bar{1}$·054	$\bar{1}$·137	$\bar{1}$·160	$\bar{1}$·036	$\bar{1}$·099	$\bar{1}$·093
0·60		$\bar{2}$·729	$\bar{2}$·549	$\bar{2}$·557	$\bar{2}$·464	$\bar{2}$·558	$\bar{2}$·490	$\bar{2}$·488	$\bar{2}$·686	$\bar{2}$·574	$\bar{2}$·646	$\bar{2}$·481	$\bar{2}$·538	$\bar{2}$·566
0·62		$\bar{3}$·850	$\bar{3}$·866	$\bar{3}$·818	$\bar{3}$·792	$\bar{3}$·885	$\bar{3}$·885	$\bar{3}$·780	$\bar{3}$·937	$\bar{2}$·039	$\bar{3}$·862	$\bar{3}$·924	$\bar{3}$·896	$\bar{3}$·893
0·64		$\bar{3}$·199	$\bar{3}$·210	$\bar{3}$·188	$\bar{3}$·248	$\bar{3}$·174	$\bar{3}$·182	$\bar{3}$·243	$\bar{3}$·057	$\bar{3}$·298	$\bar{3}$·335	$\bar{3}$·209	$\bar{3}$·265	$\bar{3}$·225
0·66		$\bar{4}$·649	$\bar{4}$·620	$\bar{4}$·587	$\bar{4}$·669	$\bar{4}$·603	$\bar{4}$·575	$\bar{4}$·614	$\bar{4}$·647	$\bar{4}$·660	$\bar{4}$·869	$\bar{4}$·740	$\bar{4}$·814	$\bar{4}$·672
0·68		$\bar{4}$·083	$\bar{5}$·939	$\bar{5}$·878	$\bar{5}$·986	$\bar{5}$·941	$\bar{5}$·804	$\bar{4}$·002	$\bar{5}$·896	$\bar{4}$·048	$\bar{4}$·115	$\bar{4}$·156	$\bar{5}$·951	$\bar{5}$·988
0·70		$\bar{5}$·525	$\bar{5}$·270	$\bar{5}$·165	$\bar{5}$·263	$\bar{5}$·235	$\bar{5}$·185	$\bar{5}$·210	$\bar{5}$·180	$\bar{5}$·378	$\bar{5}$·374	$\bar{5}$·427	$\bar{5}$·243	$\bar{5}$·305
0·72		$\bar{6}$·840	$\bar{6}$·853	$\bar{6}$·693	$\bar{6}$·720	$\bar{6}$·720	$\bar{6}$·712	$\bar{6}$·872	$\bar{6}$·706	$\bar{6}$·810	$\bar{6}$·904	$\bar{6}$·972	$\bar{6}$·740	$\bar{6}$·786
0·74		$\bar{6}$·507	$\bar{6}$·226	$\bar{6}$·110	$\bar{6}$·156	$\bar{6}$·161	$\bar{6}$·057	$\bar{6}$·125	$\bar{6}$·063	$\bar{6}$·295	$\bar{6}$·328	$\bar{6}$·527	$\bar{6}$·278	$\bar{6}$·231
0·76		$\bar{7}$·864	$\bar{7}$·684	$\bar{7}$·616	$\bar{7}$·691	$\bar{7}$·621	$\bar{7}$·599	$\bar{7}$·737	$\bar{7}$·520	$\bar{7}$·718	$\bar{7}$·820	$\bar{7}$·890	$\bar{7}$·672	$\bar{7}$·683
0·78		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Table 3(*b*) (*cont.*)

Women — Log (Relative Visibility)

Wave-length (μ)	Observer: W.V.M.L.	R.H.	E.M.C.	G.W.B.	M.R.	J.B.J.	E.N.P.	S.M.I.	M.R.S.	K.M.B.	J.E.H.	V.L.	M.J.G.
	Age: 26	21	19	19	17	25	24	20	23	21	22	20	19
0·48	$\bar{1}$·880	$\bar{1}$·909	$\bar{1}$·845	$\bar{1}$·863	$\bar{1}$·868	$\bar{1}$·829	$\bar{1}$·843	$\bar{1}$·845	$\bar{1}$·909	$\bar{1}$·873	$\bar{1}$·864	$\bar{1}$·792	$\bar{1}$·823
0·50	0·004	0·001	$\bar{1}$·960	0·017	0·003	$\bar{1}$·997	$\bar{1}$·935	$\bar{1}$·996	$\bar{1}$·996	0·036	0·088	0·000	$\bar{1}$·976
0·52	$\bar{1}$·960	$\bar{1}$·985	0·005	$\bar{1}$·949	$\bar{1}$·971	$\bar{1}$·942	0·020	$\bar{1}$·940	$\bar{1}$·958	$\bar{1}$·857	$\bar{1}$·723	$\bar{1}$·962	$\bar{1}$·986
0·54	$\bar{1}$·803	$\bar{1}$·859	$\bar{1}$·867	$\bar{1}$·834	$\bar{1}$·800	$\bar{1}$·718	$\bar{1}$·719	$\bar{1}$·757	$\bar{1}$·777	$\bar{1}$·782	$\bar{1}$·683	$\bar{1}$·733	$\bar{1}$·689
0·56	$\bar{1}$·490	$\bar{1}$·664	$\bar{1}$·586	$\bar{1}$·520	$\bar{1}$·485	$\bar{1}$·518	$\bar{1}$·590	$\bar{1}$·489	$\bar{1}$·537	$\bar{1}$·383	$\bar{1}$·503	$\bar{1}$·546	$\bar{1}$·427
0·58	$\bar{1}$·117	$\bar{1}$·180	$\bar{1}$·192	$\bar{1}$·047	$\bar{1}$·045	$\bar{1}$·011	$\bar{1}$·163	$\bar{1}$·083	$\bar{1}$·135	$\bar{1}$·233	$\bar{1}$·030	$\bar{1}$·080	$\bar{1}$·031
0·60	$\bar{2}$·465	$\bar{2}$·662	$\bar{2}$·599	$\bar{2}$·493	$\bar{2}$·398	$\bar{2}$·473	$\bar{2}$·555	$\bar{2}$·537	$\bar{2}$·573	$\bar{2}$·673	$\bar{2}$·501	$\bar{2}$·419	$\bar{2}$·512
0·62	$\bar{3}$·814	$\bar{2}$·013	$\bar{3}$·885	$\bar{3}$·842	$\bar{3}$·847	$\bar{3}$·834	$\bar{2}$·032	$\bar{3}$·895	$\bar{3}$·939	$\bar{2}$·038	$\bar{3}$·912	$\bar{3}$·836	$\bar{3}$·899
0·64	$\bar{3}$·183	$\bar{3}$·327	$\bar{3}$·295	$\bar{3}$·178	$\bar{3}$·250	$\bar{3}$·206	$\bar{3}$·390	$\bar{3}$·208	$\bar{3}$·237	$\bar{3}$·292	$\bar{3}$·341	$\bar{3}$·270	$\bar{3}$·154
0·66	$\bar{4}$·500	$\bar{4}$·794	$\bar{4}$·754	$\bar{4}$·549	$\bar{4}$·466	$\bar{4}$·377	$\bar{4}$·934	$\bar{4}$·664	$\bar{4}$·637	$\bar{4}$·661	$\bar{4}$·535	$\bar{4}$·837	$\bar{4}$·707
0·68	$\bar{5}$·825	$\bar{4}$·149	$\bar{4}$·067	$\bar{5}$·915	$\bar{5}$·964	$\bar{5}$·922	$\bar{4}$·230	$\bar{4}$·000	$\bar{4}$·027	$\bar{4}$·038	$\bar{5}$·876	$\bar{4}$·081	$\bar{4}$·026
0·70	$\bar{5}$·234	$\bar{5}$·505	$\bar{5}$·556	$\bar{5}$·310	$\bar{5}$·389	$\bar{5}$·153	$\bar{5}$·384	$\bar{5}$·315	$\bar{5}$·227	$\bar{5}$·425	$\bar{5}$·250	$\bar{5}$·381	$\bar{5}$·260
0·72	$\bar{6}$·557	$\bar{6}$·883	$\bar{6}$·922	$\bar{6}$·653	$\bar{6}$·656	$\bar{6}$·592	$\bar{6}$·719	$\bar{6}$·704	$\bar{6}$·714	$\bar{6}$·876	$\bar{6}$·745	$\bar{6}$·715	$\bar{6}$·749
0·74	$\bar{6}$·010	$\bar{6}$·305	$\bar{6}$·179	$\bar{6}$·013	$\bar{6}$·134	$\bar{6}$·096	$\bar{6}$·218	$\bar{6}$·124	$\bar{6}$·170	$\bar{6}$·227	$\bar{6}$·090	$\bar{6}$·230	$\bar{6}$·212
0·76	$\bar{7}$·463	$\bar{7}$·716	$\bar{7}$·719	$\bar{7}$·627	$\bar{7}$·597	$\bar{7}$·641	$\bar{7}$·676	$\bar{7}$·662	$\bar{7}$·639	$\bar{7}$·681	$\bar{7}$·567	$\bar{7}$·730	$\bar{7}$·728
0·78	$\bar{7}$·037	$\bar{7}$·222	$\bar{7}$·366	$\bar{7}$·130	$\bar{7}$·125	$\bar{7}$·178	$\bar{7}$·358	$\bar{7}$·196	$\bar{7}$·201	$\bar{7}$·181	$\bar{7}$·047	$\bar{7}$·186	$\bar{7}$·147

Wave-length (μ)	Observer: J.A.R.	M.W.	P.A.L.	N.S.	D.J.	P.F.	S.P.	J.M.	E.B.H.	E.H.S.	E.P.C.	A.H.	Mean
	Age: 26	20	20	25	20	20	24	22	23	22	20	27	21·8
0·48	$\bar{1}$·858	$\bar{1}$·927	$\bar{1}$·864	$\bar{1}$·864	$\bar{1}$·669	$\bar{1}$·889	$\bar{1}$·887	$\bar{1}$·773	$\bar{1}$·858	$\bar{1}$·841	$\bar{1}$·769	$\bar{1}$·819	$\bar{1}$·846
0·50	$\bar{1}$·998	$\bar{1}$·956	0·004	0·002	$\bar{1}$·951	$\bar{1}$·986	0·019	$\bar{1}$·947	$\bar{1}$·892	$\bar{1}$·977	$\bar{1}$·872	0·037	$\bar{1}$·986
0·52	$\bar{1}$·961	0·033	$\bar{1}$·948	$\bar{1}$·943	$\bar{1}$·983	$\bar{1}$·980	$\bar{1}$·915	$\bar{1}$·970	0·090	$\bar{1}$·982	0·086	$\bar{1}$·932	$\bar{1}$·963
0·54	$\bar{1}$·866	$\bar{1}$·811	$\bar{1}$·772	$\bar{1}$·715	$\bar{1}$·747	$\bar{1}$·805	$\bar{1}$·637	$\bar{1}$·718	$\bar{1}$·817	$\bar{1}$·772	$\bar{1}$·805	$\bar{1}$·780	$\bar{1}$·771
0·56	$\bar{1}$·496	$\bar{1}$·501	$\bar{1}$·491	$\bar{1}$·563	$\bar{1}$·447	$\bar{1}$·516	$\bar{1}$·366	$\bar{1}$·331	$\bar{1}$·600	$\bar{1}$·663	$\bar{1}$·480	$\bar{1}$·562	$\bar{1}$·510
0·58	$\bar{1}$·149	$\bar{1}$·150	$\bar{1}$·088	$\bar{1}$·018	$\bar{1}$·095	$\bar{1}$·098	$\bar{1}$·096	$\bar{2}$·821	$\bar{1}$·218	$\bar{1}$·202	$\bar{2}$·950	$\bar{1}$·195	$\bar{1}$·097
0·60	$\bar{2}$·647	$\bar{2}$·556	$\bar{2}$·514	$\bar{2}$·459	$\bar{2}$·715	$\bar{2}$·575	$\bar{2}$·530	$\bar{2}$·420	$\bar{2}$·768	$\bar{2}$·661	$\bar{2}$·501	$\bar{2}$·596	$\bar{2}$·552
0·62	$\bar{3}$·994	$\bar{3}$·776	$\bar{3}$·862	$\bar{3}$·821	$\bar{3}$·813	$\bar{3}$·809	$\bar{3}$·860	$\bar{3}$·758	$\bar{2}$·055	$\bar{2}$·006	$\bar{3}$·803	$\bar{3}$·845	$\bar{3}$·888
0·64	$\bar{3}$·332	$\bar{3}$·061	$\bar{3}$·308	$\bar{3}$·111	$\bar{3}$·265	$\bar{3}$·196	$\bar{3}$·226	$\bar{4}$·962	$\bar{3}$·266	$\bar{3}$·332	$\bar{3}$·078	$\bar{3}$·322	$\bar{3}$·228
0·66	$\bar{4}$·758	$\bar{4}$·427	$\bar{4}$·640	$\bar{4}$·558	$\bar{4}$·687	$\bar{4}$·515	$\bar{4}$·373	$\bar{4}$·470	$\bar{4}$·640	$\bar{4}$·733	$\bar{4}$·453	$\bar{4}$·737	$\bar{4}$·616
0·68	$\bar{4}$·131	$\bar{5}$·858	$\bar{4}$·098	$\bar{5}$·876	$\bar{4}$·001	$\bar{5}$·844	$\bar{5}$·906	$\bar{5}$·828	$\bar{4}$·012	$\bar{4}$·097	$\bar{5}$·904	$\bar{4}$·093	$\bar{5}$·991
0·70	$\bar{5}$·368	$\bar{5}$·041	$\bar{5}$·363	$\bar{5}$·321	$\bar{5}$·229	$\bar{5}$·145	$\bar{5}$·132	$\bar{5}$·159	$\bar{5}$·160	$\bar{5}$·335	$\bar{5}$·162	$\bar{5}$·659	$\bar{5}$·299
0·72	$\bar{6}$·587	$\bar{6}$·617	$\bar{6}$·794	$\bar{6}$·622	$\bar{6}$·697	$\bar{6}$·606	$\bar{6}$·635	$\bar{6}$·666	$\bar{6}$·796	$\bar{6}$·989	$\bar{6}$·672	$\bar{6}$·964	$\bar{6}$·725
0·74	$\bar{7}$·946	$\bar{6}$·057	$\bar{6}$·286	$\bar{6}$·156	$\bar{6}$·123	$\bar{6}$·094	$\bar{6}$·118	$\bar{6}$·007	$\bar{6}$·293	$\bar{6}$·390	$\bar{6}$·033	$\bar{6}$·199	$\bar{6}$·148
0·76	$\bar{7}$·695	$\bar{7}$·565	$\bar{7}$·722	$\bar{7}$·658	$\bar{7}$·532	$\bar{7}$·542	$\bar{7}$·583	$\bar{7}$·625	$\bar{7}$·785	$\bar{7}$·875	$\bar{7}$·612	$\bar{7}$·747	$\bar{7}$·655
0·78	$\bar{7}$·130	$\bar{7}$·149	$\bar{7}$·241	$\bar{7}$·057	$\bar{8}$·953	$\bar{8}$·951	$\bar{8}$·900	$\bar{8}$·981	$\bar{7}$·464	$\bar{7}$·260	$\bar{7}$·044	$\bar{7}$·483	$\bar{7}$·159

change in shape of the visibility curve, due possibly to nutritional variations, but this supposition was not upheld by direct experiments, in which six observers repeated carefully the determination of their scotopic visibility curves over a period of a year at three-monthly intervals. The mean results are shown in Table 5.

Table 4. Mean Scotopic Visibility for 50 Observers

Wavelength (μ)	Log (Sensitivity)	Wavelength (μ)	Log (Sensitivity)	Wavelength (μ)	Log (Sensitivity)
0·38	$\bar{4}$·583	0·52	$\bar{1}$·970	0·66	$\bar{4}$·644
0·40	$\bar{3}$·935	0·54	$\bar{1}$·776	0·68	$\bar{5}$·990
0·42	$\bar{2}$·988	0·56	$\bar{1}$·502	0·70	$\bar{5}$·302
0·44	$\bar{1}$·495	0·58	$\bar{1}$·095	0·72	$\bar{6}$·756
0·46	$\bar{1}$·756	0·60	$\bar{2}$·559	0·74	$\bar{6}$·190
0·48	$\bar{1}$·901	0·62	$\bar{3}$·891	0·76	$\bar{7}$·669
0·50	$\bar{1}$·983	0·64	$\bar{3}$·227	0·78	$\bar{7}$·160

Mean age : 22·4 years

Table 5

Wavelength	Log (Sensitivity)			
(μ)	November	February	May	August
0·42	$\bar{1}$·079	$\bar{1}$·062	$\bar{2}$·994	$\bar{1}$·082
0·44	$\bar{1}$·481	$\bar{1}$·538	$\bar{1}$·490	$\bar{1}$·504
0·46	$\bar{1}$·753	$\bar{1}$·734	$\bar{1}$·743	$\bar{1}$·772
0·48	$\bar{1}$·926	$\bar{1}$·932	$\bar{1}$·926	$\bar{1}$·918
0·50	$\bar{1}$·999	$\bar{1}$·976	$\bar{1}$·993	$\bar{1}$·996
0·52	$\bar{1}$·963	$\bar{1}$·988	$\bar{1}$·975	$\bar{1}$·982
0·54	$\bar{1}$·837	$\bar{1}$·819	$\bar{1}$·809	$\bar{1}$·824
0·56	$\bar{1}$·555	$\bar{1}$·513	$\bar{1}$·534	$\bar{1}$·554
0·58	$\bar{1}$·089	$\bar{1}$·063	$\bar{1}$·094	$\bar{1}$·109
0·60	$\bar{2}$·549	$\bar{2}$·495	$\bar{2}$·558	$\bar{2}$·570
0·62	$\bar{3}$·940	$\bar{3}$·879	$\bar{3}$·937	$\bar{3}$·908
0·64	$\bar{3}$·234	$\bar{3}$·183	$\bar{3}$·239	$\bar{3}$·256
0·66	$\bar{4}$·575	$\bar{4}$·526	$\bar{4}$·574	$\bar{4}$·572
0·68	$\bar{5}$·976	$\bar{5}$·892	$\bar{5}$·899	$\bar{5}$·935
0·70	$\bar{5}$·364	$\bar{5}$·276	$\bar{5}$·283	$\bar{5}$·302
0·72	$\bar{6}$·744	$\bar{6}$·641	$\bar{6}$·710	$\bar{6}$·694

There is no indication of any definite seasonal trend, and the small disagreement between the results of Tables 3(*a*) and 3(*b*) must remain for the present without an explanation. In deriving the final mean results, the figures of Tables 3(*a*) have been taken for the wavelengths shorter than that of maximum visibility, the figures of Table 3(*b*) for the longer wavelengths.

There is no evidence that there is any difference in scotopic sensitivity between the two sexes, which is a fortunate result from a practical point of view.

§4. COMPARISON OF PHOTOMETRIC AND THRESHOLD METHODS

For the reasons already discussed, the main part of the work was done with a large photometric field at the lowest practicable brightness level. To investigate the degree to which the visibility curve thus determined approximates to the ultimate scotopic curve, a few observers carried out threshold measurements

through the spectrum under the following conditions: test field circular, subtending 10° at the eye, centre at 10° below the fixation point, exposed for one second in every four seconds. The results are shown in Table 6, compared with the photometric curve for the same observers.

Table 6

	Mean Log (Sensitivity)				Mean Log (Sensitivity)		
Wavelength (μ)	Threshold method	Photometric method	Δ	Wavelength (μ)	Threshold method	Photometric method	Δ
0·40	$\bar{3}$·702	$\bar{3}$·685	−0·017	0·58	$\bar{1}$·104	$\bar{1}$·089	−0·015
0·42	$\bar{2}$·896	$\bar{2}$·961	+0·065	0·62	$\bar{3}$·895	$\bar{3}$·859	−0·036
0·44	$\bar{1}$·472	$\bar{1}$·494	+0·022	0·66	$\bar{4}$·569	$\bar{4}$·601	+0·032
0·48	$\bar{1}$·896	$\bar{1}$·894	−0·002	0·70	$\bar{5}$·232	$\bar{5}$·260	+0·028
0·50	$\bar{1}$·973	$\bar{1}$·978	+0·005	0·74	$\bar{6}$·110	$\bar{6}$·132	+0·022
0·52	0·003	$\bar{1}$·971	−0·032	0·76	$\bar{7}$·519	$\bar{7}$·598	+0·079
0·54	$\bar{1}$·900	$\bar{1}$·775	−0·125				

There is little in the figures to indicate any systematic difference between the results of threshold and photometric measurements, and it is concluded that the mean scotopic visibilities given in Table 4 represent the completely dark adapted rod mechanism of the eye to a high degree of approximation.

§ 5. COMPARISON OF THE PRESENT RESULTS WITH THOSE OF OTHER WORKERS

The chief reason for carrying out this work lay in the disagreement between the results obtained by previous workers, and the difficulty in assessing the probable value and accuracy of the various sets of data. It is therefore interesting as well as important to compare the present with previous results and to attempt to explain any discrepancies found. The more important of the earlier investigations will be considered in order of date.

(*a*) *König and Ritter*, 1892 (*König*: *Gesammelte Abhandlungen zur physiologischen Optik*, p. 144). These results are mentioned because they were the first to be obtained and are often quoted in text books, but they have little more than qualitative value because König and Ritter made no direct measurements of energy of radiation through the spectrum. Hence their results can only be expressed in terms of relative sensitivity per unit energy by making assumptions, more or less accurate, about energy distribution in the radiation from the source used and spectral transmission of their apparatus. Their results are compared with the present ones in Figure 3.

(*b*) *Abney and Watson*, 1916 (*Phil. Trans. Roy. Soc.* A, **216,** 91). The only fault to be found with these measurements is the small number of observers used, five normal observers. Otherwise the visual conditions and physical measurements appear to have been in all respects satisfactory. Comparison with the present results is made in Figure 4, from which it may be seen that agreement is good.

(*c*) *Hecht and Williams*, 1922 (*J. Gen. Physiol.*, **5,** 1). The visual conditions appear to have been good. The field of view subtended 22° at the eye, the pupil was unrestricted, and the intensity level was low. In the paper cited the field brightness is given as 2·7 times the threshold; but it is very doubtful whether photometric readings could have been made at so low a level. Possibly the

threshold referred to was measured at the fovea, or for a small test area. In any case, the exact value in ordinary units remains doubtful. In spite of good visual conditions, however, Hecht and Williams' results do not agree very well with the present results: see Figure 5. Sources of disagreement are probably to be found in that (*a*) the spectral colours were selected by a single monochromator without stray light filters, (*b*) energy measurements were made at the exit slit of the monochromator, but the filters used for cutting down intensity were

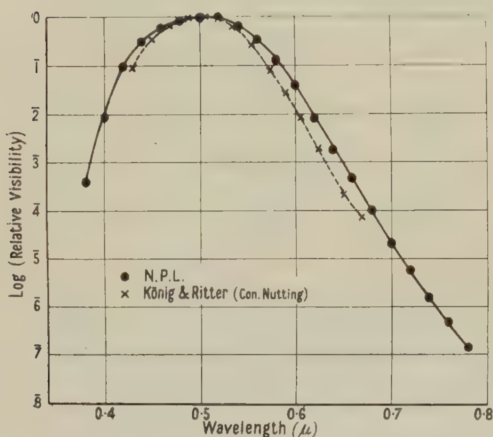

Figure 3.

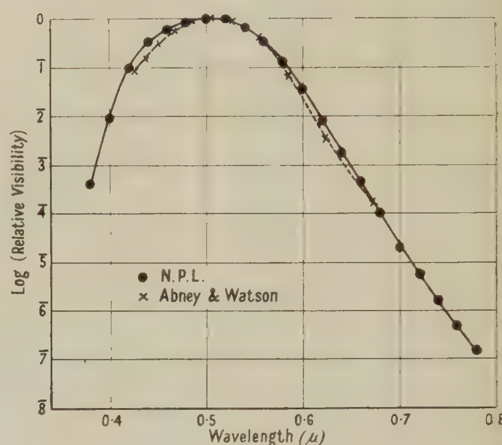

Figure 4

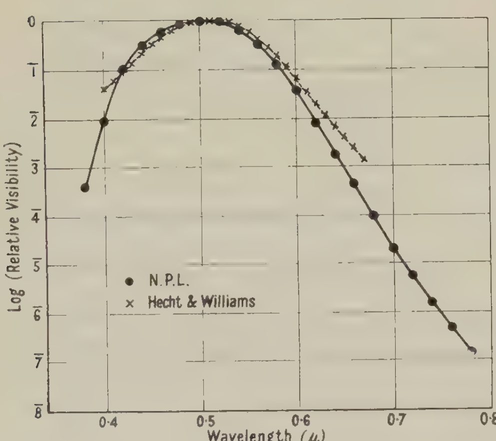

Figure 5.

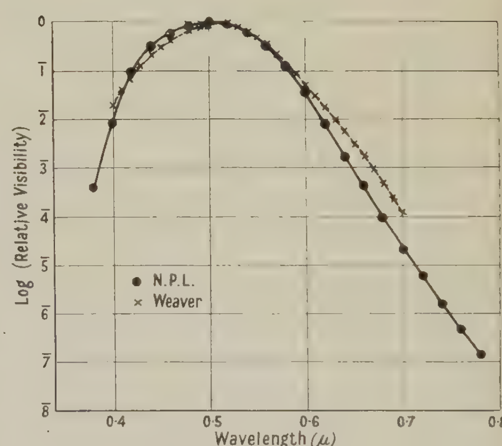

Figure 6.

measured in white light and assumed to be neutral. Reason (*a*) would give deviations from the true values in the direction shown in Figure 5 and is the likeliest explanation of the disagreement between the two sets of results.

(*d*) *Weaver*, 1937 (*J. Opt. Soc. Amer.*, **27**, 36). Weaver's results are compared with those of the present author in Figure 6. The important deviations are in the red end of the spectrum. Judging from the description of the physical conditions and measurements, these were satisfactory, but the visual conditions appear to have been unsuitable for obtaining the true scotopic visibility curve. The field was a white circular area subtending 14° at the eye with the colour

appearing in a central portion 2·6° in diameter; the brightness was $2{\cdot}1 \times 10^{-5}$ candles/ft^2 ($2{\cdot}3 \times 10^{-8}$ stilbs). Both these factors enhance foveal or cone vision, and the relatively high sensitivities in the red part of the spectrum obtained by Weaver may reasonably be attributed to them. The number (14) of observers, also, was not large.

(*e*) *Luckiesh and Taylor*, 1943 (*Illum. Eng.* (*N.Y.*), **38,** 189). The measurements by these authors, shown in Figure 7, are in striking agreement with the

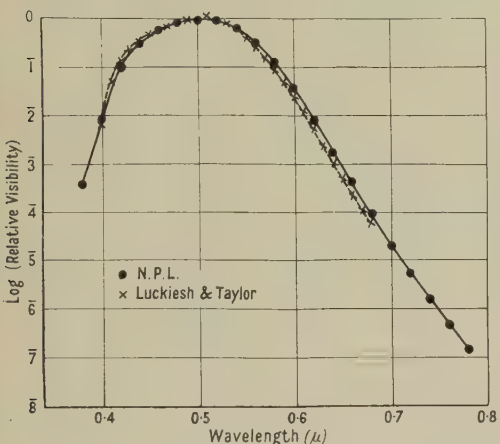

Figure 7.

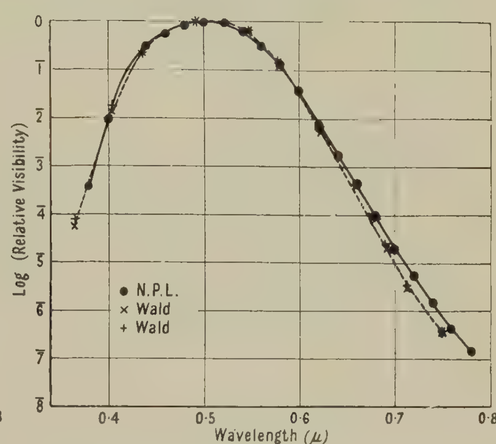

Figure 8.

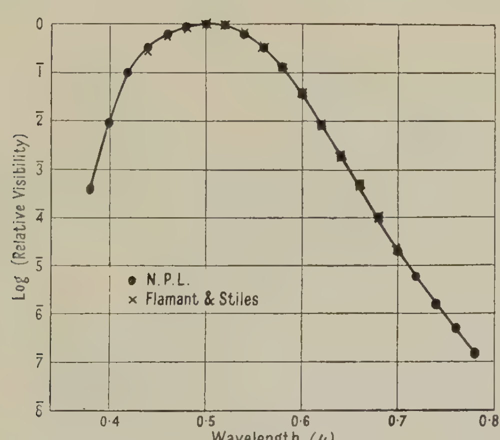

Figure 9.

present ones. Unfortunately, they are reported in the above reference with very little information about the conditions of measurement. A threshold method was used, which is good, but the field subtended only 2° at the eye and was fixated directly, conditions which enhance cone vision. There is no information about number of observers or about apparatus and the necessary physical measurements, and so it is impossible to assess the value of the work or the weight to be given to it.

(*f*) *Wald*, 1945 (*Science*, **101,** 653–658). This work was done under conditions which should give the result aimed at, the visibility curve for the completely dark adapted rod mechanism. On the physical side, monochromatic light of a

number of wavelengths was obtained by filtering from a high pressure mercury arc. On the visual side, visibility was measured as the inverse of threshold, the test field being 1° in diameter at 8° above the fixation point and exposed periodically for 1/25 second. Wald presents results for two sets of observers, one containing 22 members, the other 52, the two sets being in quite close agreement with each other. The results also agree closely with the present ones, except in the red, where they are lower: see Figure 8. This discrepancy may be due to differences in conditions of observation, to errors in energy measurements, in one investigation or the other, or possibly to a racially different batch of observers, though this is not very likely.

(*g*) *Flamant and Stiles*, 1948 (*J. Physiol.*, **107,** 187). The results of this investigation are in close agreement with those of the present paper: see Figure 9. The number of observers is, unfortunately, too small to provide a mean curve which would be a fair statistical average of the general population, but the agreement is interesting because the methods used differ substantially from any others employed heretofore. Visibility is determined as the inverse of the amount of radiation at each wavelength which is necessary to raise the threshold for green light of wavelength $0{\cdot}49\,\mu$ to ten times the absolute threshold.

§ 6. CONCLUSION

The present results appear to be in good agreement with the more reliable earlier work on scotopic visibility. Some confidence may therefore be felt in the reliability of the mean scotopic visibility function tabulated in Table 4.

ACKNOWLEDGMENTS

In work of this sort a great deal depends upon the observers, and the author desires to express his appreciation of the uniformly willing and intelligent co-operation of all concerned.

The work described above has been carried out as part of the research programme of the National Physical Laboratory and this paper is published by permission of the Director of the Laboratory.

LETTERS TO THE EDITOR

Retarding-Field Current in a Cylindrical Diode

It is known that when a negative potential $-V$ is applied to the anode of a plane thermionic diode, the current which flows as a result of the initial velocity of emission is of the form

$$i/i_0 = A \exp\,(-\boldsymbol{e}V/\boldsymbol{k}\theta) \qquad \ldots\ldots (1)$$

where θ is the cathode temperature and $\boldsymbol{k}$ is Boltzmann's constant. Therefore, if current–voltage characteristics of a plane diode are plotted on log-linear graph paper, as suggested by Rothe and Engbert (1937), the plot in the retarding-field region will be a straight line with slope determined by $\boldsymbol{k}\theta/\boldsymbol{e}$. This relation was checked in an indirectly heated diode (one half of a Mullard EB 34) in which the ratio of anode to cathode diameters is small enough to give an approximation to a plane diode, and with 6·9 volts on the heater the plot indicated $\boldsymbol{k}\theta/\boldsymbol{e} = 0{\cdot}0956$ or $\theta = 1{,}100°$ K., which is a reasonable value. But when the experiment was repeated with a diode having a thin oxide-coated filament (diode section of Mullard DAF 91), which approximates to a cylindrical system of infinite ratio, the

indicated value of θ based on equation (1) was 2,300° K. which is obviously unreasonable for an oxide-coated cathode. (The filament voltage was 1·3, as against the nominal working range for this valve of 1·4 to 1·1 volts.)

The retarding-field equation for a diode with very thin cathode should in fact be

$$i/i_0=A \exp(-eV/2k\theta), \qquad \ldots\ldots (2)$$

for the following reason. The total kinetic energy of an emitted electron may be divided into components along three cartesian axes, and in the infinite plane diode one only of these axes lies along the direction of current flow. But in a cylindrical system with infinitely thin cathode, taking the origin of coordinates at the cathode, two of the cartesian axes will be radial, so that two out of three of the velocity components are effective in conveying current. Hence the effective kinetic energy of the electrons as judged by the current–retarding-voltage characteristic is twice as high as for the plane system at the same real cathode temperature.

For a cylindrical system with finite ratio of anode to cathode diameters the contribution made by the second component can presumably be calculated by considering the relative lengths of the radial and tangential paths from a point on the cathode surface to the anode.

This difference between the effective initial velocities in plane and cylindrical diodes is likely to be relevant to the theory of the difference in noise level between plane and cylindrical diodes (Bell 1942, Weinstein 1947).

The Electrical Engineering Dept.,
The University of Birmingham,
16th March 1949.

D. A. BELL.

BELL, D. A., 1942, *J. Instn. Elect. Engrs.*, **89**, Pt. III, 207.
ROTHE, H., and ENGBERT, W., 1937, *Telefunkenröhre*, Dec.
WEINSTEIN, L. A., 1947, *Zh. Tekh. Fiz.*, **17**, 1035.

REVIEWS OF BOOKS

Theoretische Optik, by S. FLÜGGE. Pp. 124. (Hanover: Wolfenbütteler Verlagsanstalt.) No price given.

In fourteen chapters the reader is taken through the history of the development of the theory of light from Newton to quantum mechanics. This approach to the teaching of theoretical optics is more than justified in the book in question. Reference to the experimental part of the subject is kept to a bare minimum. Knowledge of vector calculus is assumed, and vector methods are used throughout—even in discussing early theories of light.

Newtonian corpuscular theory is dealt with in some detail, and the infrequently mentioned fact that this theory gives an adequate account of total internal reflection is of interest. Full attention is given to the mathematical development of the elastic solid theory, and then Maxwell's equations are formulated. A useful table is given showing the correspondence between equations and concepts in the elastic solid and electromagnetic theories.

In dealing with interference and diffraction, the theory of the circular aperture is as usual ignored. The student is consequently left unaware of the origin of the factor 0·61 in the expression for the resolving power of a lens or telescope. In the earlier part of the book the half-integral order Bessel functions are used, so there can be no excuse for omitting the circular aperture (one much used in practice) on the grounds that one of Bessel's integrals is involved. And, in fact, the standard of mathematics employed is generally quite high for a book of this kind.

The theory of dispersion, broadening of spectral lines, the optics of moving bodies and a brief introduction to quantum mechanics complete this excellent little book.

The quality of the paper used is understandably poor.

H. H. HOPKINS.

High Frequency Measuring Techniques using Transmission Lines, by E. N. PHILLIPS, W. G. STERNS and N. J. GAMARA. Pp. 64. (New York: John F. Rider Publisher, Inc., 1947). $1.50.

This brochure, by three members of the research staff of the Collins Radio Company, Cedar Rapids, Iowa, is an account of various techniques for measuring impedances and the electrical characteristics of high frequency cables which the authors have found by experience to be useful.

A special slotted coaxial transmission line with a standing wave indicator and probe is used throughout and the theory of each method, which in all cases is based on the standard transmission line formula, is given. The methods are applicable in the frequency range 100–400 Mc/s.

The brochure should prove useful to workers in the field of ultra-high-frequency measurements. L. G. H. H.

Technique of Microwave Measurements, Volume II, published by the Massachusetts Institute of Technology, Radiation Laboratory Series. Edited by C. G. MONTGOMERY. Pp. xix+939. (New York and London: McGraw Hill Publishing Co., 1947). $10.00.

Although the subject of microwaves possesses an intrinsic interest for the pure scientist, yet it was as a branch of applied science, to meet the operational needs of radar, that it showed its remarkable development during the war. A prerequisite for the success of a mass-produced radar equipment in operational use is that its performance should conform to a well defined standard with which the operator can become familiar through training and experience. It follows that the numerous component parts of such an equipment must individually conform to specified standards of performance both for this reason and also to make them readily replaceable in the event of failure. To this end, it was found necessary to develop in parallel with the various forms of microwave radar equipments, the appropriate techniques of measurement and laboratory and field test equipments for checking the performance both of a radar equipment as a whole and of its component parts individually. From these activities the important new science of Microwave Measurements developed: it is the subject of Volume II of the well known Radiation Laboratory Series on Radar. This volume, which is concerned with the range of wavelengths of 10 centimetres to 1 centimetre has been written by a team of fourteen authors, one of whom, Professor Carol G. Montgomery, is also editor of the whole work, which is of encyclopaedic proportions. Since the book gives a comprehensive account of American work in this field with occasional reference to British contributions, the devices described are all of American construction. However, the attention given throughout the book to fundamental physical principles widens its appeal and ensures that it will remain a standard work of reference despite later developments in techniques and components, as for instance in measurements at the highest available powers and improvements in crystal detectors.

After a preliminary chapter in which the subject of microwave measurements is surveyed as a whole, the remaining fourteen chapters are grouped, according to subject matter into four parts. Since no measurements are possible without sources of microwave power, Part I, comprising chapters 2 to 4, deals with power generation and measurement. The theory, construction and performance of available types of American klystron oscillators are described in chapter 2, together with their stabilized power supplies, whereas existing techniques for measuring low powers and high powers are treated in chapter 3. The remainder of Part I is devoted to microwave signal generators, receiver performance and microwave noise sources.

Part II, comprising chapters 5 to 7, covers in considerable detail the subjects of wavelength and frequency measurement. It contains much useful information about the theory and construction of resonant cavity and coaxial line wavemeters, primary and secondary frequency standards and measuring techniques. The theory and design of spectrum analysers and their practical use occupies the whole of chapter 7.

The measurement of impedance and standing waves is treated in Part III (chapters 8 to 10) which contain full discussions of the theory and technique of standing wave measurements, impedance bridges and the measurement of dielectric constants, with reference both to waveguide and coaxial line systems.

The remaining chapters, 11 to 15, which form Part IV of the book, cover the topics of attenuators (" cut off " and " resistive "), measurement of attenuation, directional couples (chapter 14) and the measurement of the radiation characteristics of microwave antennae (chapter 15) with special reference to power gain.

This volume is an outstanding contribution to the literature of microwaves and will certainly serve as a work of reference for many years for those pure physicists and those engineers who are directly concerned with the principles and applications of microwaves.

L. G. H. H.

Ionospheric Radio Propagation, National Bureau of Standards Circular 462. Pp. 209. (Superintendent of Documents, U.S. Government Printing Office, Washington, 25, D.C.). $1.00.

The scope and aim of this useful book are clearly stated in the first chapter as follows :— " The volume is not intended to be either a comprehensive treatise on the theory of wave propagation or a strictly practical handbook replete with charts and monograms to solve by rule-of-thumb all types of practical problems of radio communication. It is intended rather to set forth in simple form the physical and mathematical theory underlying the principles of radio communication by reflection from the ionosphere and to bring these principles into understandable relation with the practical problems of radio communication ".

The book is thus an introduction to ionospheric studies which is addressed both to the general student and to the communications engineer. It comprises nine chapters of which the first five survey basic principles and the existing state of knowledge of the ionosphere and are of interest to the general student, whereas the remaining four chapters which are devoted to detailed discussions of maximum usable frequency (M.U.F.), lowest required radiated power (L.R.R.F.) and the inverse quantity, lowest usable high frequency (L.U.H.F.) and to ancillary topics such as absorption in the ionosphere and radio noise, are addressed to the practical user concerned with problems of long distance communication.

The scope of the book is perhaps most simply indicated by a brief summary of the contents of the chapters. Chapter 1 is introductory and chapter 2 is a summary of the mathematical theory of radio wave propagation in an ionized medium in terms of rationalized M.K.S. units. Chapter 3 deals with techniques of measurement and discusses measurement of virtual height, ionospheric recorders, measurement of sky wave intensities, ionospheric absorption, ionospheric disturbances and radio noise.

Chapter 4 is concerned with the general structure of the ionosphere, the theory of its formation and the vertical and geographical distributions of ionization. Temporal variations both regular and sporadic are discussed in chapter 5 in relation to geomagnetic and solar phenomena. These chapters are profusely illustrated with diagrams and charts.

As already mentioned the remaining chapters comprise in effect an instructional manual in which special topics of practical importance are discussed in detail. In addition considerable attention is given to the theory and use of charts and monograms used in the prediction of the optimum parameters to be employed in radio communication by means of sky waves in specific instances.

A useful list of references to the relevant literature of the subject follows each chapter. Although the book is the product of a number of authors the writing is well coordinated and the reader is not conscious of redundancies. It forms a valuable contribution to the literature of the ionosphere at a most reasonable price. L. G. H. H.

Proteins and Life, by M. V. TRACEY, M.A. Pp. x+154. (Frontiers of Science Series. London : The Pilot Press, Ltd., 1948.) 10*s*. 6*d*.

The field of protein research is a meeting-ground of many specialized sciences, ranging from the biological to the purely physical, and it is no easy task to summarize the findings of such widely different studies. Mr. Tracey's book is addressed primarily to students and

scientists, and while it assumes no previous acquaintance with the subject, it does presuppose a knowledge of the elements of the fundamental sciences. Books of this kind may be either brief descriptions touching most corners of the field, or more detailed accounts of one or two related topics. This survey tends to conform to the former plan, but does not quite attain the end, as Mr. Tracey himself points out in the preface : " This book does not intend to cover the whole field of protein study. It is inevitable that my own interests will have coloured the interpretation of facts and influenced the selection of subjects." In this way a certain responsibility accepted by a textbook is shed. Whether this is to be regretted is, of course, debatable.

The book is divided into sections rather than chapters, with the following titles : (I) What proteins are; (II) What proteins are made of; (III) The structure of proteins; (IV) Proteins and nutrition; (V) Sources of proteins; (VI) Proteins in industry; (VII) Proteins and life; (VIII) Proteins and disease; and Conclusion.

These generalized titles are rather misleading in the light of the author's systematic approach to each particular subject. The knowledge necessary for the appreciation of the present position in different branches of protein research is presented clearly and rigorously, often in a tabular form. A nice balance is observed between the theoretical and experimental approaches. The style is pleasant and descriptions are built up by a series of well-connected short sentences. The book gains greatly by its excellent illustrations and diagrams. Above all, however, the author's lively approach to the more controversial aspects of the subject, and his interesting speculations on future lines of development, has produced a book which will be found good reading for specialists and non-specialists alike. OLGA KENNARD.

Vision and the Eye, by M. H. PIRENNE. Pp. 187. (London : The Pilot Press, Ltd., 1948.) 12*s*. 6*d*.

Dr. Pirenne states in the preface : " The eye possesses remarkable properties, among which its very high sensitivity to light is outstanding. The present book describes for the interested but non-specialist reader some of these properties and expounds part of our present knowledge of the functions of the eye." The scope of the book may be gauged from the chapter headings, which are as follows : I. The Eye and the Formation of the Retinal Image; II. The Structure of the Retina; III. Some Properties of Rods and Cones; IV. Spectral Sensitivity Curves; V. The Nervous Activity produced in the Eye by the Action of Light; VI. The minimum Energy necessary for Vision; VII. Light Quanta and the Retina; VIII. Quantum Fluctuations; IX. The Eyes and the Vision of Insects; X. Visual Acuity of Man; XI. Variation of Acuity with Light Intensity; XII. Newton's Doctrine of Colour; XIII. Normal Colour Vision; XIV. Abnormal Colour Vision; XV. Thomas Young's Theory of Colour Vision; XVI. The Two Eyes and the Brain; XVII. Physics and the Phenomena of Life.

The book will be found interesting and easy to read. The style is fluent and the print clear. The diagrams are good and the plates excellently reproduced. In all these ways the book is to be highly commended; but in others it is disappointing. Thus on page 148 is written: " . . . therefore, the present method of approach makes possible a purely objective study of the physiological response of the organism to light." The organism referred to is man himself, as Dr. Pirenne makes clear in the previous parts of the same paragraph : " The human subject of the experiment is required only to indicate whether the two halves of a photoreactive field appear identical to him or not."

In the foreword Dr. Stiles approves of Dr. Pirenne's attitude in the following words : " Two features of the exposition appeal to me strongly. These are, firstly, the very clear formulation, wherever possible in quantitative terms, of the basic notions essential to a proper understanding of the working of the eye, and secondly, the avoidance of arguments founded on introspective descriptions of sensations, which are notoriously difficult to interpret correctly."

The writer of this review cannot agree with this point of view because it suggests to the non-specialist reader, for whom the book has been written, that by using a suitable experimental layout the human subject can be converted into a species of photometer, the consequence being that the phenomenon under investigation is stated quantitatively and independently of his judgments. " Press this button when a given stimulus is seen, press

that one when it is not; or turn these knobs when two stimuli presented side by side appear different, until finally they appear to you to be the same."

It is only as the result of constant painstaking introspection that a consistent result may be finally attained. In the reviewer's opinion these ideas of Dr. Pirenne and Dr. Stiles are somewhat confusing to a non-specialist.

Some figures appear to the reviewer to be redundant. For instance, is it necessary for a book of this small size to contain Figures 9, 10, 11 and 12 ? Could not either Figure 13 or Figure 14 have been omitted ? A similar question arises about Figures 21 and 22. Lastly, is it really desirable to have Figures 23, 24, 25, 30 and 31 and Plates I, II and III ? In the reviewer's opinion some economy of space made here would have enabled a more evenly balanced account of other facets of human vision to have been presented. These suggestions are made in the hope that they may prove of value when the second edition of this book is being prepared.

H. HARTRIDGE.

CONTENTS FOR SECTION A

ABSTRACTS FOR SECTION A

The Establishment of the Absolute Scale of Temperature below 1°K., by A. H. COOKE.

ABSTRACT. Experiments are discussed establishing the relation of the absolute thermodynamic scale of temperature below 1° K. to the magnetic scale obtained by the extrapolation of Curie's law. The experimental results on iron ammonium alum, manganous ammonium sulphate, and chrome potassium alum are compared with the theoretical calculations of Onsager and Van Vleck.

A Theory of Reflectivity and Emissivity, by D. J. PRICE.

ABSTRACT. A general theory of the reflectivity and emissivity of materials is derived in a conveniently tractable form. As an example of the power of the method, it is applied to the Drude-Zener treatment of the optical properties of an ideal metal containing perfectly free electrons only. On this basis it is shown that there should exist a wide region of constant reflectivity and emissivity in the near infra-red. Krönig's theory is shown to be an approximation to the exact expression. The temperature coefficient of reflectivity is investigated and it is seen, on certain general assumptions, that it cannot change sign except in the neighbourhood of a wavelength at which the metal becomes transparent.

Disagreement with experimental observations indicates that the Drude-Zener treatment must be modified, either by a fundamental change in the dispersion formula or, more probably, by assuming that the influence of the bound electron contribution to the optical behaviour of a metal is greater than had been supposed.

The Electrical Conductivity of Germanium, by E. H. PUTLEY.

ABSTRACT. Measurements of the electrical conductivity of germanium are described. The results can be explained by the theoretical calculations of Shifrin (1944). An account is given of the relevant part of Shifrin's work. The experimental results are used to deduce the concentration of impurity centres and of thermally excited electrons and the position of the impurity levels.

Measurement of the Half-Life of ^6He, by J. E. R. HOLMES.

ABSTRACT. The half-life of ^{6}He is found to be $0{\cdot}823 \pm 0{\cdot}013$ sec. by a rotating wheel method. The statistical accuracy is considerably higher than that of previous measurements, though all results agree to within their limits of error.

The Disintegration of Carbon by Fast Neutrons, by L. L. GREEN and W. M. GIBSON.

ABSTRACT. The disintegration of ^{12}C into three α-particles by inelastic scattering of fast neutrons has been studied by the photographic plate method. Measurements made on 168 of the stars formed by this disintegration have been used to identify the reaction and to obtain values of the cross-section for the process at neutron energies between 10·8 MeV. and 14·5 MeV. No evidence was found for anisotropic scattering of the neutrons.

A Direct Method of Measuring Nuclear Spin-Lattice Relaxation Times, by L. E. DRAIN.

ABSTRACT. This direct method of measuring nuclear spin–lattice relaxation times is based on Bloch's nuclear induction experiment in which the material investigated is subjected to a magnetic field that is varied several times per second through the value for nuclear resonance. Under certain conditions a simple calculation may be made of the relation between the magnitude of the nuclear induction effect, the relaxation time, and the time intervals between successive passages through resonance. Experimental observation of this relation then determines the relaxation time directly.

Uniformly Moving Dislocations, by J. D. ESHELBY.

ABSTRACT. An expression is derived for the displacements in an isotropic elastic medium which contains an edge dislocation moving with uniform velocity c. When $c=0$ the solution reduces to that given by Burgers for a stationary edge dislocation. The energy density in the medium becomes infinite as c approaches c_2, the velocity of shear waves in the medium; this velocity therefore sets a limit beyond which the dislocation cannot be accelerated by applied stresses. The atomic structure of the medium is next partly taken into account, following the method already used by Peierls and Nabarro for the stationary dislocation. The solution found in this way differs from the one in which the atomic structure is neglected only within a region of width ζ which extends not more than a few atomic distances from the centre. ζ varies with c and vanishes when $c=c_r$, the velocity of Rayleigh waves. It becomes negative when $c_r < c < c_2$. Thus c_r rather than c_2 appears to be the limiting velocity when the atomic nature of the medium is taken into account. Since $c_r \simeq 0{\cdot}9c_2$ the difference is not of much importance.

The same method applied to a screw dislocation gives, in the purely elastic case, the expression already derived by Frank. The corresponding Peierls–Nabarro calculation shows that the width ζ is proportional to $(1-c^2/c_2^2)^{\frac{1}{2}}$. This " relativistic " behaviour is analogous to Frenkel and Kontorowa's results for their one-dimensional dislocation model.

Printed by Taylor and Francis, Ltd., Red Lion Court, Fleet Street, London E.C.4